(5) Slučajni vektori – zavisnost i nezavisnost

Pr. 5.1 Bacamo dva simetrična novčića, te

$$X := \begin{cases} 1, & \text{ako je na 1. novčiću palo } P \\ 0, & \text{———— '' ———— } G \end{cases}$$

$$Y := \begin{cases} 1, & \text{—— '' —— 2. —— '' —— } P \\ 0, & \text{—— '' —— 2. —— '' —— } G \end{cases}$$

$$\Rightarrow X \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \sim Y.$$

Uočimo, $(X,Y) : \Omega \to \mathbb{R}^2$ $\big((X,Y)(\omega) := (X(\omega), Y(\omega)) \big)$

te $(X,Y)(\omega) \in \{(i,j) : i,j \in \{0,1\}\}$, $\forall \omega \in \Omega$.

Nadalje,

$$\mathbb{P}(\underbrace{(X,Y) = (i,j)}_{=\{\omega \in \Omega :\, X(\omega)=i,\, Y(\omega)=j\}}) = \mathbb{P}(X=i, Y=j) = \frac{1}{4}, \quad \forall i,j = 0,1.$$

To zapisujemo u tablicu

$\mathbb{P}(X=0, Y=1)$

X \ Y	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

[nezavisnost]

S druge strane, ako je $Z := X$, $X \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \sim Z$, ali

$$(X,Z) \in \{(0,0), (1,1)\}$$

te $\mathbb{P}(X=0, Z=0) = \mathbb{P}(X=0) = \frac{1}{2} = \mathbb{P}(X=1, Z=1)$,

tj.

X \ Z	0	1
0	$\frac{1}{2}$	0
1	0	$\frac{1}{2}$

[potpuna zavisnost]

Dakle, distribucije slučajnih varijabli ne određuju njihovu "zajedničku distribuciju".

Def. 5.2 Ako su $X_1, \dots, X_d$ slučajne varijable na $(\Omega, \mathcal{F}, \mathbb{P})$ za $d \geq 1$, funkciju $(X_1, \dots, X_d) : \Omega \to \mathbb{R}^d$, zovemo (d-dimenzionalni) slučajni vektor. On je diskretan ako su $X_1, \dots, X_d$ diskretne slučajne varijable.

[$(X_1, \dots, X_d)$ "slučajna" točka u $\mathbb{R}^d$]

Ako je $(X_1, \dots, X_d)$ diskretan slučajni vektor i $X_i \in D_i$, $\forall i = 1, \dots, d$,

$$(X_1, \dots, X_d) \in D_1 \times \cdots \times D_d =: D$$

← opet najviše prebrojiv skup

Skup D zajedno s vjerojatnostima

$$\boxed{\mathbb{P}(X_1 = x_1, \dots, X_d = x_d), \quad x_i \in D_i, \ i = 1, \dots, d}$$

nazivamo zajednička distribucija sl. var. $X_1, \dots, X_d$ ili distribucija sl. vektora $(X_1, \dots, X_d)$.

Ako je $\boxed{d = 2}$, te $D_1 = \{a_1, \dots, a_n\}$, $D_2 = \{b_1, \dots, b_m\}$, distribucija od (X_1, X_2) može se zapisati tablicom:

$X_1 \backslash X_2$	$b_1 \cdots\cdots b_j \cdots\cdots b_m$	$\mathbb{P}(X_1 = \cdot)$
a_1		
a_2		
$\vdots$		
a_i	p_{ij}	$p_i^{(1)}$
$\vdots$		
a_n		
$\mathbb{P}(X_2 = \cdot)$	$p_j^{(2)}$	1

$$\boxed{\begin{aligned} p_{ij} &:= \mathbb{P}(X_1 = a_i, X_2 = b_j) \\ i &= 1, \dots, n, \\ j &= 1, \dots, m \end{aligned}}$$

Nadalje, za $C \subseteq \mathbb{R}^2$ vrijedi

$$\boxed{\mathbb{P}\big((X_1, X_2) \in C\big)} = \mathbb{P}\big((X_1, X_2) \in C \cap D\big)$$

$$= \mathbb{P}\Big(\bigcup_{\substack{a_i \in D_1, b_j \in D_2 \\ (a_i, b_j) \in C}} \{(X_1, X_2) = (a_i, b_j)\}\Big)$$

u parovima disj. dog.

$$= \sum \mathbb{P}\big((X_1, X_2) = (a_i, b_j)\big) =: \sum_{(a_i, b_j) \in C} p_{ij} \qquad (5.1) \quad (!)$$

Specijalno,

$$\sum_{(a_i, b_j) \in D} p_{ij} = \mathbb{P}\big((X_1, X_2) \in D\big) = 1. \qquad (5.2)$$

Također, [Kako iz $(X_1, X_2) \longrightarrow$ distribucije od X_1 i X_2?]

$$\mathbb{P}(X_1 = a_i) = \mathbb{P}\big((X_1, X_2) \in \{a_i\} \times D_2\big)$$

$$\underset{(5.1)}{=} \sum_{b_j \in D_2} p_{ij} =: p_i^{(1)}, \quad i = 1, \dots, n$$

$$\mathbb{P}(X_2 = b_j) = \dots = \sum_{a_i \in D_1} p_{ij} =: p_j^{(2)}, \quad j = 1, \dots, m$$

tablica!

tzv. marginalne distribucije varijabli X_1, X_2.

Na kraju, ako je $\boxed{g : D \to \mathbb{R}}$ f-ja, $\boxed{Y := g(X_1, X_2)}$ je diskretna slučajna varijabla. Imamo $Y \in g(D)$ te

$$\boxed{\mathbb{P}(Y = c)} = \mathbb{P}\big((X_1, X_2) \in g^{-1}(\{c\})\big)$$

$$\underset{(5.1)}{=} \sum_{(a_i, b_j) \in g^{-1}(\{c\})} p_{ij} = \boxed{\sum_{\substack{(a_i, b_j) \in D: \\ g(a_i, b_j) = c}} p_{ij}} \qquad (5.3)$$

Analogne formule vrijede u slučaju $d \in \mathbb{N}$ te kada su neki D_i-evi beskonačni skupovi, i kada je $g : D \to \mathbb{R}^k$ za $k \in \mathbb{N}$. (DZ)

Pr. 5.3 | U kutiji su 2 C, 5 B i 3 Z kuglice. Slučajno izvučemo 3 te neka je

$$X := \text{broj C kuglica},$$
$$Y := \text{broj B kuglica}.$$

Odredite (a) distr. od (X,Y), (b) distr. od X i Y, (c) $\mathbb{P}(X \geq Y)$, (d) distr. od $X+Y$.

Rj. (a) $X \in \{0,1,2\}$, $Y \in \{0,1,2,3\}$

$X \backslash Y$	0	1	2	3	$\mathbb{P}(X=\cdot)$
0	$\frac{1}{120}$	$\frac{15}{120}$	$\frac{30}{120}$	$\frac{10}{120}$	$56/120$
1	$\frac{6}{120}$	$\frac{30}{120}$	$\frac{20}{120}$	0	$56/120$
2	$\frac{3}{120}$	$\frac{5}{120}$	0	0	$8/120$
$\mathbb{P}(Y=\cdot)$	$\frac{10}{120}$	$\frac{50}{120}$	$\frac{50}{120}$	$\frac{10}{120}$	1 ✓

samo Z

$$\mathbb{P}(X=0, Y=0) = \frac{\binom{3}{3}}{\binom{10}{3}} = \frac{1}{120}$$

1B + 2Z

$$\mathbb{P}(X=0, Y=1) = \frac{\binom{5}{1}\binom{3}{2}}{120} = \frac{15}{120}$$

itd.

(b)

$$X \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{56}{120} & \frac{56}{120} & 8/120 \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{10}{120} & \frac{50}{120} & \frac{50}{120} & \frac{10}{120} \end{pmatrix}$$

(c)

$$\mathbb{P}(X \geq Y) = \left[\mathbb{P}\big((X,Y) \in \overbrace{\{(x,y) : x \geq y\}}^{=:C}\big)\right]$$
$$\overset{(5.1)}{=} \sum_{a=0}^{2} \sum_{b=0}^{a} \mathbb{P}(X=a, Y=b)$$

$$= P(X=0,Y=0) + P(X=1,Y=0) + \dots + P(X=2,Y=2)$$

$$= \frac{1}{120} \cdot (1 + 6 + 30 + 3 + 5 + 0) = \boxed{\frac{3}{8}}.$$

tabelica

(d) $X+Y = g(X,Y)$

$X \backslash Y$	0	1	2	3
0	0	1	2	3
1	1	2	3	$\cancel{4}$
2	2	3	$\cancel{4}$	$\cancel{5}$

$X+Y$

verjetnost = 0

[$X+Y = \#$ Č in B kroglic]

$\Rightarrow \quad X+Y \in \{0,1,2,3\}$ (5.3) te

$$P(X+Y=0) = P(X=0,Y=0) = \frac{1}{120}$$

$$P(X+Y=1) = P(X=1,Y=0) + P(X=0,Y=1) = \frac{1}{120}(6+15) = \frac{21}{120}$$

itd.

$$X+Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{120} & \frac{21}{120} & \frac{63}{120} & \frac{35}{120} \end{pmatrix}$$