5.1 Nezavisnost

Def. 5.4 Neka su $X_1, \dots, X_d$ diskretne slučajne varijable def. na $(\Omega, \mathcal{F}, \mathbb{P})$ t.d. je $X_i \in D_i$, $i = 1, \dots, d$. Kažemo da su $X_1, \dots, X_d$ nezavisne ako vrijedi

$$(5.4) \qquad \mathbb{P}(X_1 = x_1, \dots, X_d = x_d) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_d = x_d), \quad \forall x_i \in D_i, \ i = 1, \dots, d.$$

Nap. Uočite razliku u odnosu na događaje.

Nap. Ako su $X_1, \dots, X_d$ nezavisni, marginalne razdiobe određuju distribuciju od $(X_1, \dots, X_d)$.

Pr. 5.5 Slučajne varijable X i Y iz Pr. 5.3 nisu nezavisne ("zavisne" su) jer npr.

$$\mathbb{P}(X = 2, Y = 2) = 0 \neq \mathbb{P}(X = 2)\,\mathbb{P}(Y = 2) > 0$$

[Intuicija?]

Pr. 5.6 Bacamo kocku n puta te

$$X_i := \text{rezultat na } i\text{-toj kocki}, \quad i = 1, \dots, n$$

Imamo

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \frac{1}{6^n} = \left(\frac{1}{6}\right)^n = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i),$$

$\forall x_1, \dots, x_n \in \{1, 2, \dots, 6\}$, tj. $X_1, \dots, X_n$ su nezavisne

[to više ne provjeravamo, tj. pretp. da "očito" vrijedi.]

Prop. 5.7 | Diskretne sluč. varijable $X_1, \dots, X_d$ su nezavisne

ako

$$\boxed{\mathbb{P}(X_1 \in B_1, \dots, X_d \in B_d) = \prod_{i=1}^{d} \mathbb{P}(X_i \in B_i)}, \qquad (5.5)$$

$\forall B_i \subseteq \mathbb{R}, \; i = 1, \dots, d.$

Dokaz. | $\boxed{\Leftarrow}$ Očito jer u (5.5) možemo uzeti

$$B_i := \{x_i\}, \quad i = 1, \dots, d.$$

$\boxed{\Rightarrow}$

$$\mathbb{P}(X_1 \in B_1, \dots, X_d \in B_d) = \left[\mathbb{P}((X_1, \dots, X_d) \in B_1 \times \dots \times B_d)\right]$$

$$\overset{(5.1)}{=} \sum_{x_i \in B_i, i=1,\dots,d} \mathbb{P}(X_1 = x_1, \dots, X_d = x_d) \underset{\text{nezavisnost, tj. (5.4)}}{=} \sum_{x_i \in B_i, i=1,\dots,d} \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_d = x_d)$$

$$= \sum_{\substack{x_i \in B_i, \\ i=1,\dots,d-1}} \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_{d-1} = x_{d-1}) \cdot \underbrace{\sum_{x_d \in B_d} \mathbb{P}(X_d = x_d)}_{= \mathbb{P}(X_d \in B_d)}$$

ne ovisi o $x_1, \dots, x_{d-1}$

$$\underset{\text{analogno}}{=} \left[\sum_{\substack{x_i \in B_i, \\ i=1,\dots,d-2}} \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_{d-2} = x_{d-2})\right] \cdot \mathbb{P}(X_{d-1} \in B_{d-1}) \cdot \mathbb{P}(X_d \in B_d)$$

$$= \dots = \prod_{i=1}^{d} \mathbb{P}(X_i \in B_i) \qquad \blacksquare$$

Uočimo, ako su $X_1, \dots, X_d$ nezavisne, iz (5.5) slijedi

da su nezavisne i $X_i, i \in F$, za svaki $F \subseteq \{1, \dots, d\}$.

npr. za $F = \{1, 2, 3\}$,

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \mathbb{P}(\text{—}\,''\,\text{—}, X_4 \in \mathbb{R}, \dots, X_d \in \mathbb{R})$$

$= [(5.5)$ uz $B_i = \{x_i\}$ za $i = 1, 2, 3$ te $B_i = \mathbb{R}$, inače$]$

$= \mathbb{P}(X_1 = x_1)\,\mathbb{P}(X_2 = x_2)\,\mathbb{P}(X_3 = x_3) \cdot \underbrace{\mathbb{P}(X_4 \in \mathbb{R}) \cdots \mathbb{P}(X_d \in \mathbb{R})}_{=1}$

$\Rightarrow$ X_1, X_2, X_3 su nezavisne.

Nap. 5.8 | Ako su $X_1, \dots, X_d$ nezavisne, a $g: \mathbb{R}^m \to \mathbb{R}$ i $h: \mathbb{R}^{d-m} \to \mathbb{R}$ proizvoljne f-je, tada su sl. var.

$$g(X_1, \dots, X_m) \quad i \quad h(X_{m+1}, \dots, X_d)$$

nezavisne. $\longrightarrow$ vidi Tm. 5.6 u Sarapa, Vondraček

Nap. | Kažemo da su $(X_i, i \in \mathbb{N})$ nezavisne ako su nezavisne $X_1, \dots, X_n$, $(\forall n \in \mathbb{N})$.

Pr. 5.9 | Neka su $X \sim G(p_1)$ i $Y \sim G(p_2)$ nezavisne, $p_1, p_2 \in \langle 0, 1]$. Odredite distribuciju sluč. varijable $Z := \min\{X, Y\}$.

Rj. 1° Očito $Z \in \mathbb{N}$ jer $X, Y \in \mathbb{N}$. Uočimo,

$$\mathbb{P}(Z > n) = \mathbb{P}(X > n, Y > n) = \left[\mathbb{P}(X \in \{n+1, n+2, \dots\}, Y \in \{n+1, n+2, \dots\})\right]$$

$$= \mathbb{P}(X > n)\,\mathbb{P}(Y > n) = (1-p_1)^n (1-p_2)^n$$

nezavisnost, tj. (5.5) $\qquad X \sim G(p_1)$, $Y \sim G(p_2)$

$$= \{\underbrace{(1-p_1)(1-p_2)}_{=: q}\}^n, \quad \forall n \geq 0$$

$\{Z > n\} \subseteq \{Z > n-1\}$

$\Rightarrow \mathbb{P}(Z = n) = \mathbb{P}(\{Z > n-1\} \setminus \{Z > n\}) =$

$\mathbb{P}(Z > n-1) - \mathbb{P}(Z > n) = q^{n-1} - q^n = q^{n-1}(1-q)$, $n \geq 1$

$\Rightarrow \quad Z \sim G(1-q) = G(1-(1-p_1)(1-p_2))$.

(2°) [Intuicija!]

Neka su $X_1, X_2, \dots$, $Y_1, Y_2, \dots$ nezavisne t.d.

$$X_i \sim \begin{pmatrix} 0 & 1 \\ 1-p_1 & p_1 \end{pmatrix}, \quad Y_i \sim \begin{pmatrix} 0 & 1 \\ 1-p_2 & p_2 \end{pmatrix}, \quad \forall i \in \mathbb{N}$$

te $X := \min\{i \in \mathbb{N} : X_i = 1\}$, $Y := \min\{i \in \mathbb{N} : Y_i = 1\}$

$\underset{\text{po def.}}{\Rightarrow} \quad X \sim G(p_1), \quad Y \sim G(p_2)$ te su X i Y nezavisne.

Nadalje,

$$Z = \min\{X, Y\} = \min\{i \in \mathbb{N} : \boxed{X_i = 1 \text{ ili } Y_i = 1}\}$$

$\underset{\substack{\text{nezavisnost} \\ \text{"pokusa"}}}{\Rightarrow} \quad Z \sim G(p)$, gdje

$$p = \mathbb{P}(X_i = 1 \text{ ili } Y_i = 1)$$

$$= 1 - \mathbb{P}(X_i = 0, Y_i = 0)$$

$$\underset{X_i \text{ i } Y_i \text{ nezavisne}}{=} 1 - \mathbb{P}(X_i = 0)\mathbb{P}(Y_i = 0) \underset{\text{pretp.}}{=} 1 - (1-p_1)(1-p_2)$$

$\blacksquare$