Varijanca

Def. 4.31 | Neka je X diskretna slučajna varijabla t.d. $\mathbb{E}[|X|] < \infty$.

Varijanca od X definira se kao

$$\mathrm{Var}(X) := \mathbb{E}\Big[\underbrace{(X - \mathbb{E}[X])^2}_{\geq 0}\Big] \underline{\in [0, +\infty]} \qquad (4.21)$$

"prosječno" odstupanje sl. var. X od $\mathbb{E}[X]$

Primjer 4.32 |

$$X \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad Y \sim \begin{pmatrix} -100 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Imamo, $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, ali

$\mathbb{P}(X^2 = 1) = 1$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] = 1,$$

$$\boxed{\mathrm{Var}(Y) = \mathbb{E}[Y^2] = 100^2 \gg \mathrm{Var}(X)}$$

$\mathbb{P}(Y^2 = 100^2) = 1$

Broj $\sigma(X) := \sqrt{\mathrm{Var}(X)}$ naziva se <u>standardna devijacija</u> od X. [Statistika!]

↪ Pr. 4.32 | $\sigma(X) = 1$, $\sigma(Y) = 100$.

Prop. 4.33 | Neka je X t.d. $\mathbb{E}[|X|] < \infty$.

(i) Za $a, b \in \mathbb{R}$ vrijedi

$$\mathrm{Var}(aX + b) = a^2 \cdot \mathrm{Var}(X). \qquad (4.22)$$

(ii) $\boxed{\operatorname{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2}$ $\leftarrow$ [najčešće koristimo ovaj oblik]

(4.23)

Dokaz.

(i)
$$\operatorname{Var}(aX+b) = \mathbb{E}\left[\left(aX+b - \mathbb{E}[aX+b]\right)^2\right]$$
$$\underset{\text{Tm. 4.23 (i)}}{=} \mathbb{E}\left[\left(aX+\cancel{b} - a\,\mathbb{E}[X] - \cancel{b}\right)^2\right]$$
$$= \mathbb{E}\left[a^2\left(X - \mathbb{E}[X]\right)^2\right]$$
$$\underset{\text{Tm. 4.23 (i)}}{=} a^2\,\mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right] = a^2 \operatorname{Var}(X).$$

(ii)
$$\operatorname{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right] =$$
$$= \mathbb{E}\left[X^2 - 2X\,\mathbb{E}[X] + (\mathbb{E}[X])^2\right]$$

Nismo opravdali ak $\mathbb{E}[X^2] = +\infty$, ali vrijedi!

$$\overset{?}{=} \mathbb{E}[X^2] - \mathbb{E}\big[2X\underbrace{\mathbb{E}[X]}_{\text{const.}}\big] + (\mathbb{E}[X])^2$$
$$= \mathbb{E}[X^2] - 2\cdot\underbrace{\mathbb{E}[X]\cdot\mathbb{E}[X]}_{=(\mathbb{E}[X])^2} + (\mathbb{E}[X])^2$$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \qquad \blacksquare$$

Primjer 4.34 (Varijance poznatih razdioba)

(i) $T \sim G(p) \Rightarrow \mathbb{E}[T] = \frac{1}{p}$, te $\qquad (p \in \langle 0, 1])$

$$\mathbb{E}[\underbrace{T(T-1)}_{=g(T)}] = \sum_{k=\cancel{1}\,2}^{\infty} \underbrace{k(k-1)}_{g(k)}\, \mathbb{P}(T=k) = \sum_{k=2}^{\infty} k(k-1)\, q^{k-1}\cdot p$$

$$= p\cdot q\cdot \underbrace{\sum_{k=2}^{\infty} k(k-1)\cdot q^{k-2}}_{\left(\sum_{k=0}^{\infty} q^k\right)''} \overset{q<1}{=} p\cdot q\cdot \left((1-q)^{-1}\right)'' =$$

po q

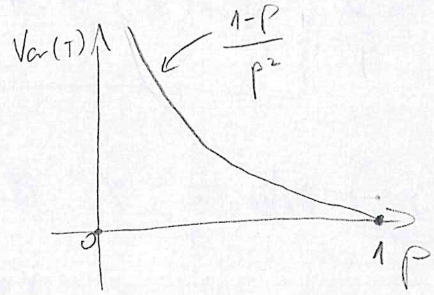
$$= p\cdot q\cdot 1\cdot 2\,\frac{1}{(1-q)^3} = 2\cdot \cancel{p}\cdot q\cdot \frac{1}{p^{\cancel{3}\,2}} = 2\cdot\frac{\overset{=1-p}{q}}{p^2} = \boxed{\frac{2}{p^2}-\frac{2}{p}}$$

$$\Rightarrow\ \mathbb{E}[T^2] = \mathbb{E}[T(T-1)+T] = \mathbb{E}[T(T-1)] + \mathbb{E}[T]$$

$$= \frac{2}{p^2}-\frac{2}{p}+\frac{1}{p} = \boxed{\frac{2}{p^2}-\frac{1}{p}}$$

$$\Rightarrow\ Var(T) = \mathbb{E}[T^2] - (\mathbb{E}[T])^2 = \frac{2}{p^2}-\frac{1}{p}-\left(\frac{1}{p}\right)^2$$

$$= \frac{1}{p^2}-\frac{1}{p} = \frac{1-p}{p^2} = \boxed{\frac{q}{p^2}}$$



Slično,

(ii) $X \sim B(n,p) \Rightarrow \boxed{Var(X) = npq}$, $n\in\mathbb{N}$, $p\in[0,1]$ ($\mathbb{E}[X]=np$)

(iii) $X \sim P(\lambda) \Rightarrow \boxed{Var(X) = \lambda = \mathbb{E}[X]}$, $\lambda > 0$ } DZ

(Pr. 4.41 i Pr. 4.42 u Šendrić, Vondraček) ∎

Nap. 4.35 Ako je $\tilde{T} \sim G_0(p) \Rightarrow T := \tilde{T}+1 \sim G(p)$

$$\Rightarrow Var(\tilde{T}) = Var(\tilde{T}+1) = Var(T) = \frac{q}{p^2}$$ ∎

Nap. 4.36 Ako je $X \sim B(p)$, tj. $X \sim \begin{pmatrix} 0 & 1 \\ q & p\end{pmatrix}$,

$$\mathbb{E}[X] = p,\quad Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\underset{X^2 = X}{=} \mathbb{E}[X] - (\mathbb{E}[X])^2$$

$$= p - p^2 = p(1-p) = \boxed{p\cdot q}$$ ∎

Nap.1 Iz (4.23) slijedi

- ako $\mathbb{E}[|X|] < \infty$ i $Var(X) < \infty \Rightarrow \mathbb{E}[X^2] < \infty$.
- ako $\mathbb{E}[X^2] < \infty \Rightarrow \mathbb{E}[|X|] < \infty$ može se pokazati

Dakle, $Var(X)$ je dobro definirana, a iz (4.23) slijedi $Var(X) < \infty$.

$$\boxed{Var(X) < \infty \Leftrightarrow \mathbb{E}[X^2] < \infty}$$

(DZ) Neka je X diskretna sluč. varijabla. Pokažite:

(a) Ako je $X \geq 0$ i $\mathbb{E}[X] = 0$, mora vrijediti

$$\mathbb{P}(X=0) = 1.$$

(b) Ako je $Var(X) = 0$, mora vrijediti

$$\mathbb{P}(X = c) = 1,$$

za neku konstantu $c \in \mathbb{R}$. Odredite i c.

(DZ) Ako vrijedi $0 \leq X \leq Y$ te $\mathbb{E}[|X|], \mathbb{E}[|Y|] < \infty$,

$$\Rightarrow \boxed{\mathbb{E}[X] \leq \mathbb{E}[Y]}. \quad (*)$$

(Primijeni Tm. 4.23 (iii) uz $b = +\infty$, na $Y - X$.)

Nap.1 $(*)$ vrijedi i kada je $\mathbb{E}[X] = +\infty$ i/ili $\mathbb{E}[Y] = +\infty$.

[iduće poglavlje]