Matematičko očekivanje (!)

[YZVB]

Def. 4.16 Neka je X diskretna sluč. varijabla t.d. $X \sim \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$. Ako je

$$\sum_{i \in I} |a_i| \cdot p_i < \infty, \quad (4.11)$$

kažemo da X ima (matematičko) očekivanje koje ("prosječna vrijednost") definiramo kao

$$\mathbb{E}[X] := \sum_{i \in I} a_i \cdot p_i \; (\in \mathbb{R}). \quad (4.12)$$

"expectation"

Nap. 4.17 Ako je $|I| < \infty$, (4.11) uvijek vrijedi, tj. X ima očekivanje.

Primjer 4.18 Ako je za $b \in [0, \frac{1}{2}]$,

$$X \sim \begin{pmatrix} -1 & 0 & 1 \\ b & \frac{1}{2} & \frac{1}{2}-b \end{pmatrix},$$

imamo

$$\mathbb{E}[X] = -1 \cdot b + 0 \cdot \frac{1}{2} + 1 \cdot \left(\frac{1}{2} - b\right) = \underline{\frac{1}{2} - 2b} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\left[\begin{array}{l} a_1 = -1, a_2 = 0, a_3 = 1 \\ p_1 = b, p_2 = \frac{1}{2}, p_3 = \frac{1}{2} - b \end{array}\right]$$

Uočimo, ako je

- $b = \frac{1}{4}$, tj. $\mathbb{P}(X=-1) = \mathbb{P}(X=1) = \frac{1}{4}$ $\longrightarrow$ $\boxed{\mathbb{E}[X] = 0}$
- $b < \frac{1}{4}$, tj. $\mathbb{P}(X=-1) < \mathbb{P}(X=1)$ $\longrightarrow$ $\boxed{\mathbb{E}[X] > 0}$
- $b > \frac{1}{4}$, tj. —"— $>$ —"— $\longrightarrow$ $\boxed{\mathbb{E}[X] < 0}$

Očekivanje $\mathbb{E}[X]$ se nekad naziva i parametar lokacije od X (tj. njene distribucije).

Nap. Ako je $X \geq 0$ (tj. $X(\omega) \geq 0, \forall \omega \in \Omega$), tada uvijek definiramo

$$\mathbb{E}[X] := \sum_{i \in I} a_i p_i \in [0, +\infty]$$

$$X \sim \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$$

$\boxed{\hookrightarrow \text{analogno za } X \leq 0}$.

Primjer 4.19 $T \sim G(p)$ za $p \in \langle 0, 1]$. $\mathbb{E}[T] = ?$

Rj.

$$\mathbb{P}(T = k) = q^{k-1} p, \quad k \in \mathbb{N} \qquad (q = 1 - p \in [0, 1\rangle)$$

$$\overset{\Rightarrow}{T \geq 0} \quad \boxed{\mathbb{E}[T]} = \sum_{k \in \mathbb{N}} k \cdot \mathbb{P}(T = k) = \sum_{k \in \mathbb{N}} k \cdot q^{k-1} p$$

$$= p \cdot \sum_{k=1}^{\infty} k \cdot q^{k-1} = \left[\sum_{k=0}^{\infty} q^k \underset{q<1}{=} \frac{1}{1-q} \,\Big|' \right]$$

$$\left[\boxed{\sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2}} \right]$$

$$= p \cdot \frac{1}{\underbrace{(1-q)^2}_{=p}} = \boxed{\frac{1}{p}}. \quad \left[\begin{array}{l} \mathbb{E}[T] \uparrow +\infty \quad \text{kada} \\ p = \text{param. uspjeha} \searrow 0 \end{array} \right]$$

Ako je $\tilde{T} \sim G_0(p)$,

$$\boxed{\mathbb{E}[\tilde{T}] \overset{DZ}{=} \frac{q}{p}}$$

Pr. 4.20 Ako je X diskretna sl. varijabla t.d. $X \in \mathbb{N}$ te

$$\boxed{\mathbb{P}(X = n) = \frac{1}{n(n+1)}}, \quad n \in \mathbb{N}$$

Imamo

$$\mathbb{E}[X] \underset{X\geq 0}{=} \sum_{n=1}^{\infty} \cancel{n} \cdot \frac{1}{\cancel{n}(n+1)} \underset{\text{harmonijski red}}{=} +\infty .$$

[X s "velikom" vjerojatnosti poprima "velike" vrijednosti] ∎

(DZ) Nađite primjer diskretne sluč. varijable koja poprima vrijednosti u $\mathbb{Z}$ te nema očekivanje; vidi Pr. 4.23 u Sarapa, Vondraček. ∎

Teorem 4.21 (!) Ako je X diskretna sluč. var. t.d. $X \in D = \{a_i : i \in I\}$ te $X \sim \begin{pmatrix} a_1 & a_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$, i $g: D \to \mathbb{R}$ f-ja, tada je

$$\mathbb{E}[g(X)] = \sum_{i\in I} g(a_i)\, p_i , \quad (4.13)$$

ako je $\sum_{i\in I} |g(a_i)|\, p_i < \infty$ ili $g(X) \geq 0$ ∎

ne moramo računati distr. od $g(X)$!

Dokaz. Neka je $g(D) = \{b_j : j \in J\}$.

Prop. 4.14. povlači ($Y = g(X)$)

$$\Rightarrow \boxed{\mathbb{P}(Y = b_j) = \sum_{\substack{i\in I:\\ g(a_i)=b_j}} p_i , \quad j \in J}$$

$$\Rightarrow \sum_{j\in J} |b_j| \cdot \mathbb{P}(Y=b_j) = \sum_j |b_j| \cdot \sum_{i: g(a_i)=b_j} p_i = \sum_j \sum_{\substack{i:\\ g(a_i)=b_j}} p_i \cdot \underbrace{|b_j|}_{=|g(a_i)|}$$

$$= \sum_{j\in J} \sum_{\substack{i\in I:\\ a_i \in g^{-1}(\{b_j\})}} p_i \cdot |g(a_i)| = \sum_{i \in I} p_i\, |g(a_i)| . \quad (4.14)$$

disjunktni skupovi te $\bigcup_{j\in J} g^{-1}(\{b_j\}) = D$

Dakle, ako je $b_j \geq 0$ $\forall j \in J$ (tj. $g(X) \geq 0$),

$$\mathbb{E}[g(X)] = \sum_{j \in J} \underset{|b_j|}{\underset{\parallel}{b_j}} \mathbb{P}(g(X) = b_j) \underset{(4.14)}{=} \sum_{i \in I} p_i g(a_i)$$
$$\in [0, +\infty].$$

Ako je $\sum_{i \in I} |g(a_i)| p_i < \infty$, (4.14) povlači da $g(X)$

ima očekivanje, te je

analogno kao u (4.14)

$$\mathbb{E}[g(X)] = \sum_{j \in J} b_j \mathbb{P}(g(X) = j) = \sum_{i \in I} p_i g(a_i)$$
$$\in [0, +\infty) \quad \blacksquare$$

Nap. 4.22 | Za $X \sim \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$ imamo

$$\mathbb{E}[|X|] \underset{g(x)=|x|}{=} \sum_{i \in I} |a_i| \cdot p_i \in [0, +\infty]$$

$$\Rightarrow \boxed{X \text{ ima očekivanje } i\ \mathbb{E}[X] \in \mathbb{R} \iff \mathbb{E}[|X|] < \infty} \quad \blacksquare$$

Tm. 4.23 | Neka je X diskretna sluč. var. t.d. $\mathbb{E}[|X|] < \infty$, te

$a, b \in \mathbb{R}$ proizvoljni. Tada vrijedi

(i) $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$

(ii) $\mathbb{P}(X = a) = 1 \Rightarrow \mathbb{E}[X] = a.$

(iii) $\mathbb{P}(a \leq X \leq b) = 1 \Rightarrow a \leq \mathbb{E}[X] \leq b.$

Dokaz | Neka je $X \sim \begin{pmatrix} a_1 & a_2 & \cdots \\ p_1 & p_2 & \cdots \end{pmatrix}$.

(i)
$$\underset{\mathbb{E}[|aX+b|]}{\underset{\parallel}{\sum_{i \in I}}} \underbrace{|a \cdot a_i + b|}_{\leq |a| \cdot |a_i| + |b|} \cdot p_i \leq |a| \cdot \underbrace{\sum_{i \in I} |a_i| p_i}_{< +\infty} + |b| \cdot \underset{< +\infty}{\overset{1}{\overset{\parallel}{\sum_{i \in I} p_i}}}$$

te slično $\mathbb{E}[aX + b] = a \cdot \sum_{i \in I} a_i p_i + b = a\,\mathbb{E}[X] + b.$

↳ Nap. | (i) vrijedi i kada samo znamo da je $X \geq 0$ te

je moguće $\mathbb{E}[X] = +\infty$ $\blacksquare$

(ii)

$$\mathbb{E}[X] = \sum_{i\in I} a_i p_i \underset{\mathbb{P}(X=x)=0,\ \forall x\neq a}{=} a\cdot 1 = a.$$

(iii)

$$\mathbb{E}[X] = \sum_{i\in I} a_i p_i \underset{a_i\notin[a,b]\ \Rightarrow\ p_i=0}{=} \sum_{\substack{i\in I:\\ a\le a_i\le b}} a_i p_i \ \begin{cases} \le b\cdot \sum_{-''-} p_i = b \\ \ge a\cdot \sum_{-''-} p_i = a \end{cases}$$

$$\sum_{i\in I} p_i = 1$$

$\square$

Nap.) Ako su $X_1,\dots,X_n$ diskretne sluč. varijable defin. na istom vjerojatnosnom prostoru $(\Omega,\mathcal{F},\mathbb{P})$ t.d. $\mathbb{E}[|X_i|]<\infty$, $\forall i=1,\dots,n$, tada je $\mathbb{E}[|\sum_{i=1}^n X_i|]<\infty$ te

$$\boxed{\mathbb{E}\Big[\sum_{i=1}^n X_i\Big] = \sum_{i=1}^n \mathbb{E}[X_i].} \qquad (4.15)$$

← iduće poglavlje

Također, (4.15) vrijedi i ako je $X_i\ge 0$, $\forall i=1,\dots,n$, te obje strane mogu biti $=+\infty$! $\square$

Pr. 4.24) $X\sim B(n,p) \Rightarrow \mathbb{E}[X]=?$

Rj.) I

$$\boxed{\mathbb{E}[X]} = \sum_{k=\cancel{0}\,1}^{n} \underbrace{\binom{n}{k} p^k q^{n-k}}_{=\mathbb{P}(X=k)}\cdot k$$

$$= \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{\cancel{k}(k-1)!} p^k q^{n-k}\cdot \cancel{k}$$

$$= n\cdot p \sum_{k=1}^{n} \frac{(n-1)\cdots(n-k+1)}{(k-1)!} p^{k-1} q^{n-k}$$

$$\underset{l=k-1}{=} n\cdot p \sum_{l=0}^{n-1} \underbrace{\frac{(n-1)\cdots(n-l)}{l!}}_{=\binom{n-1}{l}} p^l q^{(n-1)-l}$$

$$\underset{\text{Binomni teorem}}{=} n\cdot p\,(\underbrace{p+q}_{=1})^{n-1} = \boxed{n\cdot p}$$

(II) Ako je $A_i = \{$ uspjeh u i-tom pokusu $\}$, $i = 1, \dots, n$

(bolje) $\Rightarrow$ $X := \sum_{i=1}^{n} \mathbb{1}_{A_i} \sim B(n, p)$, te $\mathbb{1}_{A_i} \sim \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$.

$(A_i)_i$ nezavisni te $P(A_i) = p, \forall i$

$$(4.15) \Rightarrow \mathbb{E}[X] = \mathbb{E}\Big[\sum_{i=1}^{n} \mathbb{1}_{A_i}\Big] \overset{(4.15)}{=} \sum_{i=1}^{n} \mathbb{E}[\mathbb{1}_{A_i}]$$

$$= \sum_{i=1}^{n} \underbrace{(0 \cdot q + 1 \cdot p)}_{= p, \forall i} = \underline{n \cdot p} \qquad \blacksquare$$

Tm. 4.25 | Neka je X diskretna sluč. varijabla t.d. $\boxed{X \in \mathbb{N}_0}$.

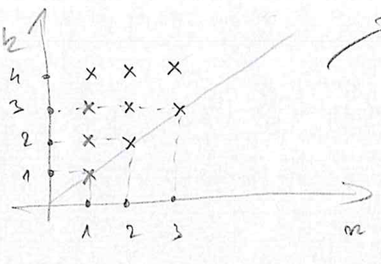
Tada je

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=0}^{\infty} P(X > n) \in [0, +\infty] \qquad (4.16) \qquad \blacksquare$$

Dokaz | $\sum_{n=1}^{\infty} P(X \geq n) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(X = k)$

ne ovisi o n

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} P(X = k)$$



$$= \sum_{k=\cancel{1}0}^{\infty} k \cdot P(X = k) = \mathbb{E}[X] \qquad \blacksquare$$

Primjer 4.26 | Za $T \sim G(p)$ imamo

$$P(\underbrace{T > n}_{\text{neuspjeh u prvih } n \text{ pokusa}}) = q^n, \quad \forall n \in \mathbb{N}_0$$

$$\Rightarrow \mathbb{E}[T] = \sum_{n=0}^{\infty} P(T > n) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} = \frac{1}{p} \qquad \blacksquare$$

$T \in \mathbb{N} \subseteq \mathbb{N}_0$