Nejednakosti

[Bitno u vjerojatnosti!]

Tm. 4.37 Neka je X diskretna slučajna varijabla i $h : \mathbb{R} \to [0, +\infty)$ f-ja. Tada za sve $a > 0$ vrijedi

$$\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}[h(X)]}{a} \qquad (4.24)$$ ∎

Dokaz Pretp. da je $X \sim \begin{pmatrix} a_1 & a_2 & \dots \\ p_1 & p_2 & \dots \end{pmatrix}$.

$$\Rightarrow \mathbb{E}[h(X)] = \sum_{i \in I} p_i \, h(a_i) = \sum_{i : h(a_i) \geq a} \text{-''-} + \cancel{\sum_{i : h(a_i) < a} \text{-''-}}$$

$$\geq \sum_{i : h(a_i) \geq a} p_i \, h(a_i) \geq \sum_{i : h(a_i) \geq a} p_i \cdot a$$

$h(a_i) \geq 0, \forall i$

$$= a \cdot \mathbb{P}(h(X) \geq a).$$

(!)

Prop. 4.38 Neka je X diskretna sluč. varijabla. Za sve $a > 0$, vrijedi

- $\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[|X|]}{a}$ (Markovljeva nejednakost)
- $\mathbb{P}(|X - \mathbb{E}[X]| \geq a) \leq \frac{Var(X)}{a^2}$. (Čebiševljeva nejednakost)

Dokaz Slijedi iz Tm. 4.37 uz $h(y) = |y|$, odn. $h(y) = |y - \mathbb{E}[X]|^2$. ∎

Primjer 4.39 Neka je $X_n \sim B(n, p)$ i $\varepsilon > 0$ proizvoljan.

$$\mathbb{P}(|X_n - \mathbb{E}[X_n]| \geq \overbrace{\varepsilon \cdot \mathbb{E}[X_n]}^{=:a}) \leq \frac{Var(X_n)}{\varepsilon^2 \cdot \mathbb{E}[X_n]^2} =$$

Čebiševljeva nejednakost

$$= \frac{n \cdot p \cdot q}{\varepsilon^2 \cdot (np)^2} = \frac{q}{\varepsilon^2 \cdot p} \cdot \frac{1}{n} \xrightarrow[p]{n\to\infty} 0 . \quad [\text{"Large deviation result"}]$$

$\boxed{\text{ako je } p \text{ fiksan}}$

Dakle,

$$\mathbb{P}\left((1-\varepsilon)\,\mathbb{E}[X_n] < X_n < (1+\varepsilon)\,\mathbb{E}[X_n] \right) = \mathbb{P}\left(| X_n - \mathbb{E}[X_n] | < \varepsilon \cdot \mathbb{E}[X_n] \right)$$

$$\xrightarrow[p \text{ fiksan}]{n\to\infty} 1 , \quad \forall \varepsilon > 0$$

DZ Pretp. da je $Var(X) < \infty$. Pokažite da za sve $c > 0$ vrijedi

$$\mathbb{P}\left(| X - \mathbb{E}[X] | \geq c \cdot \sigma(X) \right) \leq \frac{1}{c^2} .$$

npr. za $\boxed{c = 3}$, $\boxed{\frac{1}{c^2} = \frac{1}{9} \approx 0.11}$

[$\mathbb{E}[X]$ i $\sigma(X)$ nam dakle nešto govore o X !]