2) Uvjetna vjerojatnost i nezavisnost [Od sada imamo svoju notaciju]

[Bitno!]

Primjer] Bacamo dvije kocke

$$\to \Omega = \{(i,j) : i,j \in \{1,\dots,6\}\}.$$

Ako je

$$A = \{\text{zbroj je } 4\} = \{(i, 4-i) : i = 1,2,3\}$$

$$\Rightarrow \quad \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{3}{36} = \frac{1}{12}$$

Ako dodatno znamo da se dogodio

$$B = \{\text{na prvoj kocki pao broj } 2\}$$
$$= \{(2,j) : j = 1,2,\dots,6\},$$

kako se mijenja vjerojatnost za A? ...

$B \leftarrow$ "novi" Ω

Budući da su sni ishodi

$$\overbrace{(2,1), \boxed{(2,2)}, (2,3), (2,4), (2,5), (2,6)}^{B}$$

$\swarrow$ A

jednako vjerojatni, a samo 1 je povoljan za A

$$\Rightarrow \quad \mathbb{P}(A|B) := \text{"vjerojatnost od A uz uvjet B"}$$

$$= \frac{1}{6} \quad \left(> \frac{1}{12} = \mathbb{P}(A) \right)$$

Uočimo,

$$\mathbb{P}(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A\cap B|/|\Omega|}{|B|/|\Omega|} = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}.$$

<u>Defin. 2.1</u>| Neka je $(\Omega, \mathcal{F}, \mathbb{P})$ vjerojatnosni prostor te $B \in \mathcal{F}$ t.d. $\mathbb{P}(B) > 0$. Za sve $A \in \mathcal{F}$ defin.

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \qquad (2.1)$$

te $\mathbb{P}(A \mid B)$ zovemo vjerojatnost od A uz uvjet B (ili uz uvjet B) ■

Iz (2.1) odmah slijedi

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\, \mathbb{P}(B) \quad (!) \qquad (2.2)$$

<u>Nap. 2.2</u>| Ako je $\mathbb{P}(B) = 0$, $\mathbb{P}(A \mid B)$ nije definirana, ali svejedno pišemo (2.2), tj. $\mathbb{P}(A \mid B)\mathbb{P}(B) := 0 = \mathbb{P}(A \cap B)$ $(\mathbb{P}(A \cap B) \leq \mathbb{P}(B) = 0)$ ■

Iz (2.2), $\forall A, B \in \mathcal{F}$ imamo

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \quad (!) \qquad (2.3) \\ &= \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\, \mathbb{P}(B^c) \end{aligned}$$

[FPV]

Općenito, za familiju događaja $(H_i)_{i \in I}$ pri čemu je $I = \{1, 2, \dots, n\}$ ili $I = \mathbb{N}$, kažemo da je <u>potpun sistem događaja</u> (PSD) ako je

- $H_i \cap H_j = \emptyset$, $\forall i \neq j$ (u parovima disjunktni)
- $\bigcup_{i \in I} H_i = \Omega$.

[u skripti se traži $\mathbb{P}(H_i) > 0$, $\forall i$, ali to nije potrebno zbog Nap. 2.2.]

<u>Prop. 2.3</u>| (Formula potpune vjerojatnosti)

Za sve $A \in \mathcal{F}$ i bilo koji PSD $(H_i)_{i\in I}$ vrijedi

$$\mathbb{P}(A) = \sum_{i \in I} \mathbb{P}(A|H_i)\, \mathbb{P}(H_i) \qquad (2.4) \quad (\text{"FPV"})$$

Dokaz.

$$\mathbb{P}(A) = \mathbb{P}(A \cap \Omega) = \mathbb{P}(A \cap (\bigcup_{i\in I} H_i))$$

$$= \mathbb{P}(\bigcup_{i\in I} (A \cap H_i)) \underset{\uparrow}{=} \sum_{i\in I} \mathbb{P}(A \cap H_i)$$

σ-aditivnost (ili konačna adit.) jer su $(A\cap H_i)_{i\in I}$ paroviмa disj.

$$= \underset{(2.2)}{} \sum_{i\in I} \mathbb{P}(A|H_i)\mathbb{P}(H_i) \qquad \blacksquare$$

DZ) Ako je $A \in \mathcal{F}$, i $(H_i)_{i\in I}$ familija u parovima disj. događaja t.d. $A \subseteq \bigcup_{i\in I} H_i$ (umjesto $\bigcup_{i\in I} H_i = \Omega$), pokažite da i dalje vrijedi (2.4).

[Rj. gledaj $\tilde{H}_i := H_i$ za $i \in I$ te $\tilde{H}_0 := (\bigcup_{i\in I} H_i)^c$

$\Rightarrow$ $\tilde{H}_0, \tilde{H}_i, i \in I$ je PSD te je $\mathbb{P}(A|\tilde{H}_0)\mathbb{P}(\tilde{H}_0) = \mathbb{P}(A \cap \tilde{H}_0) = 0$]

Primjer 2.4 | Igrači A, B, C redom bacaju kocku sve dok ne dobiju broj 6 (vidi poglavlje (1)). Ako je

$A_1 = \{A$ je prvi dobio $6\}$, te $a := \mathbb{P}(A_1)$

$B_1 = \{B$ — " — $\}$, te $b := \mathbb{P}(B_1)$

$C_1 = \{C$ — " — $\}$, te $c := \mathbb{P}(C_1)$

Odredite a, b i c.

Rj. $a = \mathbb{P}(A_1) = [\, H_1 = \{A$ dobio 6 u 1. bacanju$\}, H_2 = H_1^c \,]$

$\nwarrow$ PSD

$$\underset{FPV}{=} \underbrace{\mathbb{P}(A_1 | H_1)}_{= \dots = 1} \underbrace{\mathbb{P}(H_1)}_{= \frac{1}{6}} + \mathbb{P}(A_1 | H_1^c) \underbrace{\mathbb{P}(H_1^c)}_{= \frac{5}{6}}$$

$$= \frac{1}{6} + \frac{5}{6} \cdot \underbrace{\mathbb{P}(A_1 | H_1^c)}_{} = \frac{1}{6} + \frac{5}{6} \mathbb{P}(C_1) = \frac{1}{6} + \frac{5}{6} \cdot c.$$

ako 1. igrač nije odmah dobio 6 "postaje" 3. igrač po redu

$$\Rightarrow c = \mathbb{P}(C_1) = ?$$

Uočimo

$$C_1 \subseteq \{ \text{u prva dva bacanja nije pala } 6\text{ica} \} =: D,$$

te $D = (D \cap E_1) \cup (D \cap E_2) =: H_1' \cup H_2'$, gdje je

$E_1 = \{ \text{u svom 1. bacanju, C je dobio } 6 \}$,

$E_2 = E_1^c$.

Budući da je $H_1' \cap H_2' = \emptyset$ i $A \subseteq H_1' \cup H_2'$,

$$c = \mathbb{P}(C_1) \underset{FPV}{=} \mathbb{P}(C_1 | H_1') \mathbb{P}(H_1') + \underbrace{\mathbb{P}(C_1 | H_2')} \mathbb{P}(H_2')$$

- $\mathbb{P}(C_1 | H_1') = \mathbb{P}(C_1 | \underbrace{\text{u prva dva bacanja} \neq 6, \text{ u 3. } 6}) = 1$
- $\mathbb{P}(H_1') = \mathbb{P}(\quad) = \frac{5^2 \cdot 1}{6^3} \left[= \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \right] = \frac{25}{216}$
- $\mathbb{P}(C_1 | H_2') = \mathbb{P}(C_1 | \text{u prva tri bacanja} \neq 6)$

  $= [\text{kao da krećemo ispočetka}]$

  $= \mathbb{P}(C_1) = c$
- $\mathbb{P}(H_2') = \frac{5^3}{6^3} = \frac{125}{216}$.

$$\Rightarrow c = \frac{25}{216} + \frac{125}{216} \cdot c \quad \Rightarrow \quad c = \frac{25}{91}$$

Dakle,

$$a = \frac{1}{6} + \frac{5}{6}c = \dots = \frac{36}{91}. \longleftarrow$$ [usporedi s rješenjem u Poglavlju ① !?]

(DZ) Pokažite da je

$$b = \frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot c.,$$

Specijalno, $b = \frac{30}{91}$ $(=1-a-c)$ ∎

Nap. 1 (!) Nekad nas zanima $\mathbb{P}(A|B)$ te ju računamo po formuli (2.1), ali često "znamo" $\mathbb{P}(A|B)$ te ju koristimo kako bi računali npr. $\mathbb{P}(A)$ ili $\mathbb{P}(A\cap B)$ ∎

Tm. 2.5 (Bayesova formula) (!)
Neka je $A \in \mathcal{F}$ t.d. $\mathbb{P}(A)>0$ te $(H_i)_{i\in I}$ konačan ili prebrojiv niz događaja t.d. $A \subseteq \bigcup_{i\in I} H_i$ i $H_i\cap H_j=\emptyset, \forall i\neq j$. Tada je za sve $j\in I$,

$$\mathbb{P}(H_j|A) = \frac{\mathbb{P}(A|H_j)\,\mathbb{P}(H_j)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|H_j)\,\mathbb{P}(H_j)}{\sum_{i\in I}\mathbb{P}(A|H_i)\,\mathbb{P}(H_i)} \qquad (2.5)$$

Dokaz.

$$\mathbb{P}(H_j|A) \overset{(2.1)}{=} \frac{\mathbb{P}(H_j\cap A)}{\mathbb{P}(A)} \overset{(2.2)}{=} \frac{\mathbb{P}(A|H_j)\,\mathbb{P}(H_j)}{\mathbb{P}(A)} \overset{(2.4)}{=} \frac{\mathbb{P}(A|H_j)\,\mathbb{P}(H_j)}{\sum_{i\in I}\mathbb{P}(A|H_i)\,\mathbb{P}(H_i)}$$ ∎

Primjer 2.6 ([illegible])
Rijetka bolest pogađa 1 u svakih 100 000 osoba. Test koji imate je dosta pouzdan: ako imate bolest, test će biti pozitivan s vjer. 0.95, a ako nemate, bit će (lažno) pozitivan s vjer. 0.005. Test je pokazao da imate bolest, koliko je vjer. da ju zaista imate?

Rj. Neka je

- $H_1 = \{\text{imate bolest}\} \rightarrow \mathbb{P}(H_1) = \frac{1}{100\,000}$
- $H_2 = \{\text{nemate bolest}\} \rightarrow \mathbb{P}(H_2) = 1 - \mathbb{P}(H_1) = \frac{99\,999}{100\,000}$

$\Big\}$ PSD

- $T = \{\text{test je pozitivan}\} \rightarrow \mathbb{P}(T|H_1) = 0.95,\ \mathbb{P}(T|H_2) = 0.005$

Tražimo $\mathbb{P}(H_1|T)$!

$$\underline{\underline{\mathbb{P}(H_1|T)}} \overset{(2.5)}{=} \frac{\mathbb{P}(T|H_1)\mathbb{P}(H_1)}{\mathbb{P}(T|H_1)\mathbb{P}(H_1) + \mathbb{P}(T|H_2)\mathbb{P}(H_2)} = \frac{0.95 \cdot \frac{1}{100\,000}}{0.95 \cdot \frac{1}{100\,000} + 0.005 \frac{99\,999}{100\,000}}$$

$$= \frac{0.95}{0.95 + 499.995} \approx 0.002 = 0.2\,\% \quad \Big(> \underline{\underline{\mathbb{P}(H_1)}}\Big)$$

ali

$\Rightarrow$ Je li test beskoristan? Ne nužno! U praksi se testiraju $\underline{\underline{\text{samo}}}$ osobe kod kojih se sumnja da imaju bolest, na temelju nekih drugih indikatora, a za njih je "$\mathbb{P}(H_1)$" $\underline{\underline{\text{puno veći}}}$ nego u $\underline{\underline{\text{općenitoj}}}$ populaciji.
$\hookrightarrow$ (a $\mathbb{P}(H_2)$ $\underline{\underline{\text{puno manji}}}$)

<u>Ostala svojstva uvjetne vjerojatnosti</u>

Ako su $A_1, \ldots, A_n \in \mathcal{F}$ događaji, tada je

"chain rule"

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_2 \cap A_1) \cdots \mathbb{P}(A_n|A_{n-1} \cap \cdots \cap A_1) \qquad (2.6)$$

ukoliko je $\mathbb{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$ (pa su sve gornje uvjetne vjerojatnosti dobro definirane). (DZ)

<u>Primjer 2.7</u> U kutiji se nalazi m B i n C kuglica, te na slučajan način izvučemo $k \leq m$ kuglica, jednu za drugom (bez vraćanja). Kolika je vjerojatnost da smo izvukli $\underline{\underline{\text{samo}}}$ B kuglice?

Rj. Neka je B_k traženi događaj, te

$A_i := \{$ izvukli B kuglicu u i-tom izvlačenju$\}$, $i=1,\dots,k$

$$\Rightarrow \mathbb{P}(B_k) = \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_k) \overset{(2.6)}{=} \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\cdots\mathbb{P}(A_k|A_{k-1}\cap\dots\cap A_1)$$

- $\mathbb{P}(A_1) = \frac{n}{2n} = \frac{1}{2}$ $\left[\text{također, } \mathbb{P}(A_i) = \frac{1}{2}, \forall i\right]$
- $\mathbb{P}(A_2|A_1) = [$u kutiji ostalo $n-1$ B i n C kuglica$]$

$$= \frac{n-1}{2n-1}$$

- $\mathbb{P}(A_3|A_2\cap A_1) = \frac{n-2}{2n-2}$

$\vdots$

- $\mathbb{P}(A_k|A_1\cap\dots\cap A_{k-1}) = \frac{n-(k-1)}{2n-(k-1)} = \frac{n-k+1}{2n-k+1}$

$$\Rightarrow \mathbb{P}(B_k) = \frac{n}{2n}\cdot\frac{n-1}{2n-1}\cdots\frac{n-k+1}{2n-k+1}$$

Za sve $B\in\mathcal{F}$ t.d. $\mathbb{P}(B)>0$ definiramo f-ju $\mathbb{P}_B : \mathcal{F}\to\mathbb{R}$ s

$$\mathbb{P}_B(A) := \mathbb{P}(A|B), \quad A\in\mathcal{F} \qquad (2.7)$$

<u>Prop. 2.8</u> | Neka je $B\in\mathcal{F}$ t.d. $\mathbb{P}(B)>0$. Tada je $\mathbb{P}_B$ <u>vjerojatnost</u> na $(\Omega,\mathcal{F})$.

<u>Dokaz</u> | (A1) $\mathbb{P}_B(A) = \mathbb{P}(A|B) \geq 0$, $\forall A\in\mathcal{F}$ ✓

(A2) $\mathbb{P}_B(\Omega) = \mathbb{P}(\Omega|B) = \frac{\mathbb{P}(\Omega\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$ ✓

(A3) Ako su $(A_j)_{j\in\mathbb{N}}$ u parovima disjunktni događaji iz $\mathcal{F}$,

tada su i $(A_j\cap B)_{j\in\mathbb{N}}$ —— " ——.

$$\Rightarrow \mathbb{P}_B\left(\bigcup_{j=1}^{\infty} A_j\right) = \frac{\mathbb{P}\left(\left(\bigcup_{j=1}^{\infty} A_j\right)\cap B\right)}{\mathbb{P}(B)} = \frac{\mathbb{P}\left(\bigcup_{j=1}^{\infty}(A_j\cap B)\right)}{\mathbb{P}(B)} =$$

$$\sum_{j=1}^{\infty} \frac{P(A_j \cap B)}{P(B)} = \sum_{j=1}^{\infty} P_B(A_j).$$

∎

Dakle, $P_B(\cdot) = P(\cdot \mid B)$ ima sva svojstva "standardne" vjerojatnosti, npr.

- $P(A^c \mid B) = 1 - P(A \mid B)$, ili (2.8)
- $A \cap C = \emptyset \Rightarrow P(A \cup C \mid B) = P(A \mid B) + P(C \mid B)$ (2.9)

Nap. | Budući da je $P_B(A) = P_B(A \cap B)$, $\forall A \in \mathcal{F}$ (DZ), u (2.9) i sličnim tvrdnjama, dovoljno je da vrijedi $(A \cap B) \cap (C \cap B) = \emptyset$. ∎

Nadalje, za $A, C \in \mathcal{F}$ t.d. $P_B(A) = P(A \mid B) > 0$ (tj. $P(A \cap B) > 0$), vrijedi

$$\boxed{P_B(C \mid A) = P(C \mid A \cap B)} \qquad (2.10) \qquad (DZ)$$

Specijalno,

$$P(A \cap C \mid B) = P(A \mid B)\, P(C \mid A \cap B), \qquad (2.11)$$

te ako je $(H_i)_{i \in I}$ konačan ili prebrojiv niz u parovima disjunktnih događaja t.d. za $A \in \mathcal{F}$, $A \subseteq \bigcup_{i \in I} H_i$,

$$\begin{aligned} P(A \mid B) &= P_B(A) = [\underline{\text{FPV}} \text{ za } P_B\,!] \\ &= \sum_{i \in I} P_B(A \mid H_i)\, P_B(H_i) \\ &\overset{(2.10)}{=} \sum_{i \in I} P(A \mid H_i \cap B)\, P(H_i \mid B) \end{aligned} \qquad (2.12)$$

[Dovoljno je da je $A \cap B \subseteq \bigcup_{i \in I} (H_i \cap B)$.]

Primjer 2.05 | Na temelju iskustva i statistike, osiguravajuća kuća smatra da postoje dvije vrste ljudi:

- oni skloni nezgodama → vjerojatnost da imaju nezgodu u bilo kojoj godini je 0.4 (neovisno o nezgodama u drugim godinama)*
- oni koji nisu skloni nezgodama → —— " —— 0.2 . —— " ——

Ako pretp. da je 30% ljudi u populaciji koji uzimaju osiguranje sklono nezgodama, kolika je vjerojatnost da će novi osiguranik imati nezgodu u 1. godini? Ako je imao nezgodu u 1. godini, kolika je vjerojatnost da će je imati i u 2. godini?

Rj.: Neka je

• $A = \{\text{osiguranik je sklon nezgodama}\}$

$$\xrightarrow{\text{zadano}} \mathbb{P}(A) = \underline{0.3}$$

$$\to \mathbb{P}(A^c) = \mathbb{P}(\{\text{osig. nije sklonu nezg.}\}) = \underline{0.7}$$

• $A_i = \{\text{osiguranik ima nezgodu u } i\text{-toj godini}\},\ i=1,2$

$$\xrightarrow{\text{zadano}} \bullet\ \mathbb{P}(A_1|A) = 0.4 = \mathbb{P}(A_2|A),$$

$$\text{te}\quad 0.4 = \mathbb{P}_A(A_2) \overset{*}{=} \mathbb{P}_A(A_2|A_1) \overset{(2.10)}{=} \mathbb{P}(A_2|A_1\cap A)$$

$$\bullet\ \mathbb{P}(A_2|A^c) = 0.2 = \mathbb{P}(A_2|A^c) = \mathbb{P}(A_2|A^c\cap A_1)$$

reći ćemo da su A_1 i A_2 uvjetno nezavisni uz uvjet A

1° Zanima nas

$$\underline{\underline{\mathbb{P}(A_1)}} \overset{FPV}{=} \mathbb{P}(A_1|A)\mathbb{P}(A) + \mathbb{P}(A_1|A^c)\mathbb{P}(A^c) = 0.4\cdot 0.3 + 0.2\cdot 0.7 = \underline{\underline{0.26}} \quad (= \mathbb{P}(A_2)).$$

2° Zanima nas

$$\mathbb{P}(A_2|A_1) \overset{(2.12)}{=} \underbrace{\mathbb{P}(A_2|A_1\cap A)}_{=\mathbb{P}(A_2|A)=0.4}\mathbb{P}(A|A_1) + \underbrace{\mathbb{P}(A_2|A_1\cap A^c)}_{=\mathbb{P}(A_2|A^c)=0.2}\mathbb{P}(A^c|A_1)$$

$$\bullet\ \mathbb{P}(A|A_1) \overset{\text{Bayes}}{=} \frac{\mathbb{P}(A_1|A)\mathbb{P}(A)}{\mathbb{P}(A_1)} = \frac{0.4\cdot 0.3}{0.26} = \frac{6}{13} \approx 0.46$$

$$\to \underline{\underline{\mathbb{P}(A^c|A_1)}} \underset{(2.8)}{=} 1 - \mathbb{P}(A|A_1) = \underline{\underline{\frac{7}{13}}} \approx 0.54$$

$$\to \underline{\underline{\mathbb{P}(A_2|A_1)}} \underset{(\star)}{=} 0.4\cdot 0.46 + 0.2\cdot 0.54 = \underline{\underline{0.292}}$$

uočimo)

$$\mathbb{P}(A_2 \mid A_1) > \mathbb{P}(A_2)$$

jer je $\boxed{\mathbb{P}(A \mid A_1) \approx 0.46 > 0.3 = \mathbb{P}(A)}$ (pa i

$\mathbb{P}(A^c \mid A_1) < \mathbb{P}(A^c)$)