1.3 Konačan vjerojatnosni prostor i Laplaceov model

Pretp. da je $|\Omega| = n \ (<+\infty)$, tj. $\Omega = \{\omega_1, \dots, \omega_n\}$.

Ako pretp. da je $\{\omega_i\} \in \mathcal{F}, \forall i = 1, \dots, n$, tada je nužno $\mathcal{F} = \mathcal{P}(\Omega)$.
(DZ)

Prop. 1.8 Neka je $|\Omega| = n$ te $\mathcal{F} = \mathcal{P}(\Omega)$.

(a) Svaka vjerojatnost $\mathbb{P}$ na $(\Omega, \mathcal{F})$ je u potpunosti određena s vrijednostima $\mathbb{P}(\{\omega_i\})$, $i = 1, \dots, n$.

(b) Za svaki vektor $= (p_1, \dots, p_n) \in \mathbb{R}^n$ t.d. $p_i \geq 0, \forall i$ te $\sum_{i=1}^{n} p_i = 1$, postoji (i jedinstvena je) vjerojatnost $\mathbb{P}$ na $(\Omega, \mathcal{F})$ t.d. vrijedi

$$\mathbb{P}(\{\omega_i\}) = p_i, \quad i = 1, \dots, n.$$

Dokaz (a) Neka je $\underline{\underline{p_i := \mathbb{P}(\{\omega_i\})}}$, $i = 1, \dots, n$. Za svaki $A \in \mathcal{F}$ imamo

$$A = \{\omega_{i_1}, \dots, \omega_{i_k}\},$$

za neki $k \leq n$, pa je nužno

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{l=1}^{k} \{\omega_{i_l}\}\right) \underset{\substack{\text{konačna} \\ \text{aditivnost} \\ \text{od } \mathbb{P}}}{=} \sum_{l=1}^{k} \underbrace{\mathbb{P}(\{\omega_{i_l}\})}_{p_{i_l}} = \underline{\sum_{l=1}^{k} p_{i_l}}. \quad (*)$$

(b) Iz $(*)$ slijedi da je jedini kandidat za $\mathbb{P}$ funkcija $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ definirana s

$$\mathbb{P}(A) := \sum_{l=1}^{k} p_{i_l}, \quad \text{za } A = \{\omega_{i_1}, \dots, \omega_{i_k}\} \in \mathcal{F}, \quad (**)$$

(uz $\mathbb{P}(\emptyset) := 0$).

Pokažimo da je $\mathbb{P}$ zaista vjerojatnost:

$$(A1) \quad \mathbb{P}(A) = \sum_{l=1}^{k} p_{i_l} \geq 0 \quad \text{jer } p_i \geq 0, \ \forall i=1,\dots,n$$

za $A = \{\omega_{i_1}, \dots, \omega_{i_k}\} \in \mathcal{F}$ proizvoljan,

$$(A2) \quad \mathbb{P}(\Omega) = \mathbb{P}(\{\omega_1, \dots, \omega_n\}) \underset{(**)}{=} \sum_{i=1}^{n} p_i = 1.$$

(A3) Budući da je $|\Omega| = n < +\infty$, ako su $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, u parovima disjunktni može postoji $k \in \mathbb{N}$ t.d.

$$A_j = \emptyset, \quad \text{za } j \geq k+1,$$

pa je

$$\mathbb{P}\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mathbb{P}\Big(\bigcup_{j=1}^{k} A_j\Big).$$

Ako pokažemo da je $\mathbb{P}$ konačno adit., iz $\mathbb{P}(\emptyset) = 0$ imamo

$$\mathbb{P}\Big(\bigcup_{j=1}^{k} A_j\Big) = \sum_{j=1}^{k} \mathbb{P}(A_j) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

Pokažimo konačnu aditivnost. Neka su $A_1, \dots, A_k \in \mathcal{F}$ u parovima disjunktni. Tada možemo zapisati

$$\bigcup_{l=1}^{k} A_l = \{\underbrace{\omega_{i_1}, \dots, \omega_{i_{k_1}}}_{=A_1}, \underbrace{\omega_{i_{k_1+1}}, \dots, \omega_{i_{k_1+k_2}}}_{=A_2, \text{ itd.}}, \dots\}$$

$$\Rightarrow \mathbb{P}\Big(\bigcup_{l=1}^{k} A_l\Big) \underset{(**)}{=} \underbrace{p_{i_1} + \dots + p_{i_{k_1}}} + \underbrace{p_{i_{k_1+1}} + \dots + p_{i_{k_1+k_2}}} + \dots$$

$$\underset{(**)}{=} \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots + \mathbb{P}(A_k) \quad \blacksquare$$

Nap.) Prop. 1.8. vrijedi i kada je Ω prebrojivo beskonačan (bez dokaza) $\blacksquare$

Laplaceov model

Ako je $\Omega = \{\omega_1, \dots, \omega_n\}$, te su svi elem. dog. jednako vjerojatni, tj. $\mathbb{P}(\{\omega_i\}) = p \in [0,1]$, $\forall i = 1, \dots, n$, imamo

$$1 = \sum_{i=1}^{n} \mathbb{P}(\{\omega_i\}) = n \cdot p,$$

tj. $p = \frac{1}{n}$. Za proizvoljan $A = \{\omega_{i_1}, \dots, \omega_{i_k}\} \in \mathcal{F}$ je nužno

$$\mathbb{P}(A) = \sum_{\ell=1}^{k} \underbrace{\mathbb{P}(\{\omega_{i_\ell}\})}_{=\frac{1}{n}} = \frac{k}{n} = \frac{|A|}{|\Omega|}.$$

kon. adit.

tzv. Laplaceov model.

Primjer) Bacamo dvije kocke, kolika je vjerojatnost da

(a) pokažu isti broj?

(b) suma im je 7?

Rj.) Za Ω možemo uzeti

$$\Omega = \{(i,j) : i,j = 1, \dots, 6\} = \{1, 2, \dots, 6\}^2,$$

te su svi $\omega \in \Omega$ jednako vjerojatni $\Rightarrow$ Laplaceov model.

a) Ako je A traženi događaj,

$$A = \{(i,i) : i = 1, \dots, 6\}$$

$$\Rightarrow \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{6^2} = \frac{1}{6}$$

b) — " — B — " —,

$|B| = 6$

$$B = \{(i, 7-i) : i = 1, \dots, 6\} \Rightarrow \mathbb{P}(B) = \frac{6}{6^2} = \frac{1}{6}.$$

Nap.] Bilo je bitno veličinosti kocke, u suprotnome je

$$\Omega = \{(i,j) : 1 \le i \le j \le 6\},$$

manji od brojeva na kockama — veći –||–

ali nisu svi $\omega \in \Omega$ jednako vjerojatni, npr.

$$\mathbb{P}(\{(1,2)\}) = \frac{2}{36}, \quad \mathbb{P}(\{(1,1)\}) = \frac{1}{36},$$

pa ne možemo koristiti Laplaceov model. ▨

u ovom slučaju,

$$\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_{i=1}^{6} \{(i,i)\}\Big) = \sum_{i=1}^{6} \underbrace{\mathbb{P}(\{(i,i)\})}_{=\frac{1}{36}} = \frac{6}{36} = \frac{1}{6}$$

Pr. 1.11] U kutiji je n bijelih i n crnih kuglica, te na slučajan način izvučemo iz kutije 2 kuglice, jednu za drugom (bez vraćanja).

(a) Odredite Ω.

(b) Odredite vjerojatnost da su izvučene kuglice različite boje.

(c) Odredite

$$p_n := \mathbb{P}(\{\text{kuglice iste boje}\}),$$

te izračunajte $\lim_{n\to\infty} p_n$.

Rj. (a) Jedna mogućnost je pretp. da su kuglice označene brojevima $1, 2, \dots, 2n$ (npr. $1, \dots, n$ su crne), te uzeti

$$\Omega = \{(i,j) : i,j \in \{1,2,\dots,2n\},\ i \ne j\}.$$

prva izvučena kuglica — druga –||–

ne možemo istu kuglicu izabrati dva puta

Očito su svi $\omega \in \Omega$ jednako vjerojatni → Laplaceov model.
$(|\Omega| = 2n(2n-1))$

Nap.] Alternativno, $\Omega = \{BB, BC, CB, CC\}$, ali nisu svi $\omega \in \Omega$ jednako vjerojatni. Koji imaju veću vjerojatnost? ▨

(uočite, ovo ne ovisi o odnosu broja C i B kuglica) → (b)

(b) Ako je A traženi događaj,

$$A = \{1. \text{ kuglica } C,\ 2.\ B\} \cup \{1. \text{ kuglica } B,\ 2.\ C\}$$
$$=: \quad A_1 \quad \cup \quad A_2$$

Iz $A_1 \cap A_2 = \emptyset$, slijedi

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) = \frac{|A_1|}{|\Omega|} + \frac{|A_2|}{|\Omega|} = \left[A_1 = \{(i,j) : \overbrace{i \in \{1, \dots, n\}}^{\text{crne}}, \overbrace{j \in \{n+1, \dots, 2n\}}^{\text{bijele}}\}\right]$$

$$= \frac{n \cdot n}{2n(2n-1)} + \frac{n \cdot n}{2n(2n-1)} = \cancel{2} \cdot \frac{n}{\cancel{2}(2n-1)} = \frac{n}{2n-1} \quad \left(> \frac{n}{2n} = \frac{1}{2},\ \forall n \in \mathbb{N} \right)$$

(c) Očito,

$$p_n = \mathbb{P}(A^c) = 1 - \mathbb{P}(A) \underset{(b)}{=} 1 - \frac{n}{2n-1} = \frac{n-1}{2n-1} \quad \left(< \frac{1}{2},\ \forall n \in \mathbb{N} \right),$$

ali

$$\lim_{n \to \infty} p_n = \frac{1}{2}.$$

Nap. Pretp. da su kuglice izvučene istovremeno

$$\Longrightarrow \Omega = \{\{i,j\} : i,j \in \{1, \dots, 2n\},\ i \neq j\}$$

Opet su svi $\omega \in \Omega$ jednako vjerojatni, te je

$$|\Omega| = \binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1).$$

Sada je $\quad A = \{\underline{\underline{\{i,j\}}} : i \in \{1, \dots, n\}, j \in \{n+1, \dots, 2n\}\}$

$$|A| = \underset{\substack{\uparrow \\ 1 \text{ bijela}}}{\binom{n}{1}} \cdot \underset{\substack{\uparrow \\ 1 \text{ crna}}}{\binom{n}{1}} = n^2,$$

te

$$\mathbb{P}(A) = \frac{n^2}{n(2n-1)} = \frac{n}{2n-1} \qquad (\text{isto, naravno!})$$

↳ općenito, biramo onaj pristup (tj. Ω) s kojim lakše rješavamo problem ∎

Pr. 1.12 | Uz pretp. kao u prethodnom primjeru:

(a) $\mathbb{P}(\underbrace{\{1.\text{ kuglica je } B\}}_{B_1}) = ?$,

(b) $\mathbb{P}(\underbrace{\{2.\text{ kuglica je } B\}}_{B_2}) = ?$

Rj. (a) Događaj B_1 ovisi samo o prvom izvlačenju pa možemo gledati samo ishode tog izvlačenja, tj. gledati

$$\Omega_1 := \{1, 2, \ldots, 2n\}$$

$\Rightarrow$ [opet Laplaceov model]

$$\Rightarrow \quad \mathbb{P}(B_1) = \frac{n}{2n} = \frac{1}{2}.$$

(b) [Rješenje?]

Ako gledamo samo 2. izvlačenje, očito svaka od $2n$ kuglica ima jednaku vjerojatnost da bude izvučena:

$$\Omega_2 = \{1, 2, \ldots, 2n\}$$

$\Rightarrow$ opet Laplaceov model $\Rightarrow \mathbb{P}(B_2) = \frac{n}{2n} = \frac{1}{2}$.

Alternativno, ako je

$$\Omega = \{(\underset{1.}{i}, \underset{2.\text{ kuglica}}{j}) : i, j \in \{1, \ldots, 2n\},\ i \neq j\},$$

$$|B_2| = |\{1.\text{ kuglica } B,\ 2.\ B\}| + |\{1.\text{ kuglica } C,\ 2.\ B\}|$$

$$= n(n-1) + n \cdot n = n(2n-1)$$

$$\Rightarrow \mathbb{P}(B_2) = \frac{\cancel{n(2n-1)}}{2\cancel{n}\cancel{(2n-1)}} = \frac{1}{2} \quad \blacksquare$$

Primjer 1 Igrači A, B i C bacaju kocku tim redoslijedom. Svaki igrač ispada iz igre kada prvi puta dobije šesticu. Odredite

$$\mathbb{P}(\underbrace{\{A \text{ je ispao } i\text{-ti po redu}\}}_{=:A_i}), \quad i=1,2,3.$$

Rj.) Ispostavlja se da je za Ω najbolje uzeti malo veći prostor, tj. pretp. da svaki igrač nastavlja bacanje kocke i nakon što dobije šesticu

$$\rightarrow \Omega = \{(x_1, x_2, x_3, x_4, \dots) : x_i \in \{1,\dots,6\}, \forall i \in \mathbb{N}\}$$

$\hookrightarrow$ ($x_1, x_4, \dots$ su bacanja 1. igrača; $x_2, x_5, \dots$ — " — 2. igrača; $x_3, x_6, \dots$ — " — 3. igrača)

(1.) $\mathbb{P}(A_1) = ?$

Ako je $D_k = \{$Igrač A je ispao nakon točno k svojih bacanja$\}$, $k \in \mathbb{N}$, imamo

$$D := \bigcup_{k=1}^{\infty} D_k = \{\text{Igrač } A \text{ je nekad ispao}\}$$

te

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 \cap D) + \underbrace{\mathbb{P}(\underbrace{A_1 \cap D^c}_{=\emptyset})}_{=0}$$

$$= \mathbb{P}\left(A_1 \cap \left(\bigcup_{k=1}^{\infty} D_k\right)\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} (A_1 \cap D_k)\right)$$

inače odmah

$$= \sum_{k=1}^{\infty} \mathbb{P}(A_1 \cap D_k)$$

$(D_k)_{k\geq 1}$ u par. disj.
$\Rightarrow (A_1 \cap D_k)_{k \geq 1}$ — " —

Za $k \geq 1$, D_k (pa onda i $A_1 \cap D_k$) ovisi samo o prvih $\underbrace{3(k-1)+1}_{=3k-2}$ bacanja.

Ako gledamo $A_1 \cap D_k$ kao događaj u

svi mogući ishodi u prvih $3k-2$ bacanja

$$\tilde{\Omega} = \{(x_1, \dots, x_{3k-2}) : x_j \in \{1,\dots,6\}, \forall j\}$$

budući da su svi $\omega \in \tilde{\Omega}$ jednako vjerojatni, imamo

kao $\subseteq \tilde{\Omega}$

$$\mathbb{P}(A_1\cap D_k) = \frac{|A_1\cap D_k|}{|\tilde{\Omega}|}$$

Uočimo,

$$A_1\cap D_k = \{(x_1,\dots,x_{3k-2}) : x_{3k-2} = 6,\ x_j\in\{1,\dots,5\} \text{ za } j\neq 3k-2\}$$

$$\Rightarrow |A_1\cap D_k| = \underbrace{5^{3k-3}}_{\text{bilo što osim 6 u prvih } 3k-3 \text{ bacanja}}\cdot \overset{\text{6ica}}{1} \Rightarrow \boxed{\mathbb{P}(A_1\cap D_k) = \frac{5^{3k-3}}{6^{3k-2}} = \left(\frac{5}{6}\right)^{3k-3}\frac{1}{6} = \left(\frac{5}{6}\right)^{3(k-1)}\cdot\frac{1}{6}}$$

Dakle,

$$\boxed{\mathbb{P}(A_1)} = \sum_{k=1}^{\infty}\mathbb{P}(A_1\cap D_k) = \sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{3(k-1)}\frac{1}{6}$$

$$= \frac{1}{6}\sum_{m=0}^{\infty}\left(\frac{5^3}{6^3}\right)^m = \frac{1}{6}\,\frac{1}{1-\frac{5^3}{6^3}} = \frac{1}{\cancel{6}}\,\frac{6^{\cancel{3}\,2}}{6^3-5^3}$$

za $|x|<1$, $\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$

$$= \frac{6^2}{6^3-5^3} = \boxed{\frac{36}{91}}$$

Izračunajmo $\mathbb{P}(A_2)$: [Dr. pristup]

$$A_2\cap D_k = [\underbrace{A_2\cap D_k\cap B_1}_{\{B \text{ ispao prvi}\}}]\cup[\underbrace{A_2\cap D_k\cap C_1}_{\{C \text{ ispao prvi}\}}]$$

- $A_2\cap D_k\cap B_1 = \{(x_1,\dots,x_{3k-2}) : x_{3k-2}=6$, <u>barem</u> jedna 6ica u $\underbrace{x_2,x_4,\dots,x_{3k-4}}_{k-1 \text{ bacanja}}$, niti jedna 6ica u $\underbrace{x_3,x_6,\dots,x_{3k-3}}_{k-1}\}$

$$\Rightarrow |A_2\cap D_k\cap B_1| = 1\cdot\underbrace{(6^{k-1}-5^{k-1})}_{\text{barem jedna 6ica u } k-1 \text{ bacanja}}\cdot 5^{k-1}\cdot 5^{k-1} \quad \Big|\ \{\underbrace{x_1,x_4,\dots,x_{3k-5}}_{k-1}\}$$

$$= (6^{k-1}-5^{k-1})\,5^{2k-2}$$

• analogno

$$|A_2 \cap D_k \cap C_1| = \text{isto} = (6^k - 5^k)\, 5^{2k-1}$$

$$\Rightarrow \quad \mathbb{P}(A_2 \cap D_k) = \frac{2 \cdot (6^{k-1} - 5^{k-1})\, 5^{2k-2}}{6^{3k-2}}, \quad \forall k \geq 1$$

$$\Rightarrow \quad \boxed{\mathbb{P}(A_2)} = \sum_{k=1}^{\infty} \mathbb{P}(A_2 \cap D_k) = 2 \sum_{k=1}^{\infty} \frac{6^{k-1} 5^{2k-2}}{6^{3k-2}} - 2 \sum_{k=1}^{\infty} \frac{5^{3k-3}}{6^{3k-2}}$$

DZ

$$= 2 \cdot \sum_{k=1}^{\infty} \frac{(6 \cdot 5^2)^{k-1}}{(6^3)^{k-1} \cdot 6} - 2 \sum_{k=1}^{\infty} \frac{(5^3)^{k-1}}{(6^3)^{k-1} \cdot 6}$$

$$= \frac{2}{6} \cdot \frac{1}{1 - \frac{\cancel{6} \cdot 5^2}{6^{\cancel{3}\,2}}} - \frac{2}{6} \cdot \frac{1}{1 - \frac{5^3}{6^3}} =$$

$$= \frac{1}{3} \left(\frac{6^2}{6^2 - 5^2} - \frac{6^3}{6^3 - 5^3} \right)$$

$$= \frac{6^2}{3} \left(\frac{1}{11} - \frac{6}{91} \right)$$

$$= 8 \left(\frac{91 - 66}{1001} \right) = \frac{♡ \cdot 25}{1001} = \boxed{\frac{300}{1001}}.$$

$$\mathbb{P}(A_3) \overset{DZ}{=} \frac{305}{1001}.$$

?!

Uočimo,

$$\underset{\frac{396}{1001}}{\underset{\parallel}{\mathbb{P}(A_1)}} > \mathbb{P}(A_3) > \mathbb{P}(A_2)$$

■

[Ostali zadaci na vježbama.]