Zad. 7.8 | Koristeći CGT pokažite da je

$$\lim_{n\to+\infty} \overbrace{e^{-n} \cdot \sum_{k=0}^{n} \frac{n^k}{k!}}^{=(*)} = \frac{1}{2}.$$

(Rj.) • ako $S_n \sim P(n) \Rightarrow \mathbb{P}(S_n = k) = \frac{n^k}{k!} e^{-n}, \forall k \in \mathbb{N}_0$

$$\Rightarrow \boxed{\mathbb{P}(S_n \leq n) = (*)}$$

• s predavanja znamo, ako su $(X_i)_{i=1,2,\dots}$ nezav. t.d. $X_1 \sim P(1)$, tada je $S_n := \sum_{i=1}^{n} X_i \sim P(\underbrace{1+1+\dots+1}_{n}) = P(n)$.

$\Rightarrow$ CGT $(\underbrace{\mathbb{E}[X_1]}_{\mu} = \underbrace{Var(X_1)}_{=\sigma^2=1} = 1)$

$$\mathbb{P}(S_n \leq n) = \mathbb{P}\Big(\frac{S_n - n \cdot \overset{\mu}{1}}{\sqrt{n} \cdot \underset{\sigma}{1}} \leq 0\Big) \xrightarrow{CGT} \Phi(0) = \frac{1}{2}$$

Zad. 9.13* | Koristeći slabi ZVB, za sve $x, a, b \in (0,1)$, $a<b$ i $x \neq a, b$, odredite

$$\underbrace{\lim_n \sum_{\substack{r \in \mathbb{N}: \\ an < r < bn}} \binom{n}{r} x^r (1-x)^{n-r}}_{=: C_n(x)}.$$

(R) Ako je $S_n \sim B(n, x)$, $\forall n$

$$\Rightarrow C_n(x) = \mathbb{P}(S_n \in (an, bn)) = \mathbb{P}\left(\frac{S_n}{n} \in (a,b)\right)$$

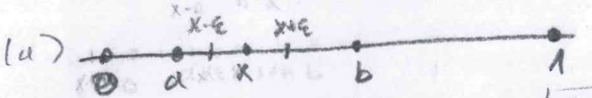
$S_n = X_1 + \dots + X_n$ $\forall n$ za $(X_i)_{i=1,2,\dots}$ njd. t.d. $X_1 \sim \begin{pmatrix} 0 & 1 \\ 1-x & x \end{pmatrix}$

$$\underset{SZVB}{\Rightarrow} \frac{S_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1] = x, \text{ tj. } \underbrace{\mathbb{P}\left(\left|\frac{S_n}{n} - x\right| \geq \varepsilon\right)}_{= \mathbb{P}\left(\frac{S_n}{n} \notin (x-\varepsilon, x+\varepsilon)\right)} \to 0, \ \forall \varepsilon > 0.$$

Tvrdimo: (a) ako $x \in (a,b) \Rightarrow C_n(x) \longrightarrow 1$, $n \to \infty$

(b) ako $x \notin (a,b) \Rightarrow C_n(x) \longrightarrow 0$, $n \to \infty$

(a)



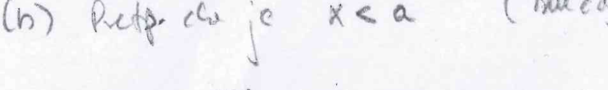
Uzmimo $\varepsilon > 0$ t.d. $\boxed{\varepsilon < \min\{b-x, x-a\}}$

$$\Rightarrow 0 \leq \underbrace{\mathbb{P}\left(\frac{S_n}{n} \notin (a,b)\right)}_{1 - C_n(x)} \leq \mathbb{P}\left(\frac{S_n}{n} \notin (x-\varepsilon, x+\varepsilon)\right) \xrightarrow[SZVB]{n\to\infty} 0$$

$(x-\varepsilon, x+\varepsilon) \subseteq (a,b)$

$$\Rightarrow C_n(x) \xrightarrow[n\to\infty]{} 1$$

(b) Pretp. da je $x < a$ (slučaj $x > b$ ide analogno)



Uzmimo $\varepsilon > 0$ t.d. $\boxed{x + \varepsilon < a}$

$$\Rightarrow 0 \leq \underbrace{\mathbb{P}\left(\frac{S_n}{n} \in (a,b)\right)}_{C_n(x)} \leq \mathbb{P}\left(\frac{S_n}{n} \notin (x-\varepsilon, x+\varepsilon)\right) \xrightarrow[SZVB]{n\to\infty} 0$$

$(x-\varepsilon, x+\varepsilon) \cap (a,b) = \emptyset$

$$\Rightarrow C_n(x) \xrightarrow[n\to\infty]{} 0$$