7 U dvoslojnom modelu za slučaj $U_1 = -U_3'$ i $k^2 = \lambda^2$, nađite ovisnost $\psi_3'(\psi_1')$ za totalno stabilne valove ($\delta = 0$).

Rj: diferencijalne jednadžbe za 2LM:

$$1)\left(\frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x}\right)\frac{\partial^2 \psi_m}{\partial x^2} + \beta \frac{\partial \psi_m}{\partial x} + U_T \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_T}{\partial x^2}\right) = 0$$

$$2)\left(\frac{\partial}{\partial t} + U_m \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi_T}{\partial x^2} - 2\lambda^2 \psi_T\right) + \beta \frac{\partial \psi_T}{\partial x} + U_T \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi_m}{\partial x^2} + 2\lambda^2 \psi_m\right) = 0$$

- probna rješenja: $\left.\begin{aligned}\psi_m &= Ae^{ik(x-ct)} \\ \psi_T &= Be^{ik(x-ct)}\end{aligned}\right\} \Rightarrow$ ubacimo u 1) i 2)

$$\Rightarrow (1)\ \left[(c - U_m)k^2 + \beta\right]A - U_T k^2 B = 0$$

$$(2)\ -U_T(k^2 - 2\lambda^2)A + \left[(c - U_m)(k^2 + 2\lambda^2) + \beta\right]B = 0$$

$\Rightarrow$ dobije se disperzijska relacija:

$$c = U_m - \frac{\beta(k^2+\lambda^2)}{k^2(k^2+2\lambda^2)} + \sqrt{\underbrace{\frac{\beta^2\lambda^4}{k^4(k^2+2\lambda^2)^2} - \frac{U_T^2(2\lambda^2 - k^2)}{k^2+2\lambda^2}}_{\delta}}$$

- zadatak kaže $U_1 = -U_3 \Rightarrow U_m = \frac{U_1+U_3}{2} = 0$; $U_T = \frac{U_1 - U_3}{2} = U_1 = -U_3$

- znamo: $\psi_m = \frac{\psi_1' + \psi_3'}{2}$; $\psi_T = \frac{\psi_1' - \psi_3'}{2}$

$$\Rightarrow \frac{\psi_m}{A} = \frac{\psi_T}{B} \Rightarrow \frac{\psi_1' + \psi_3'}{\cancel{2}A} = \frac{\psi_1' - \psi_3'}{\cancel{2}B} \Rightarrow \psi_1' + \psi_3' = \frac{A}{B}\psi_1' - \frac{A}{B}\psi_3'$$

$$\Rightarrow \left(1 + \frac{A}{B}\right)\psi_3' = \left(\frac{A}{B} - 1\right)\psi_1' \Rightarrow \psi_3' = \frac{\frac{A}{B} - 1}{\frac{A}{B} + 1}\psi_1' \Rightarrow \boxed{\psi_3' = \frac{A-B}{A+B}\psi_1'}$$

- kada odredimo koeficijente A i B, zadatak je riješen!
- da bi valovi bili stabilni, ne smije postojati c_{im}, a to znači da δ mora biti ≥ 0! za totalno stabilne $\Rightarrow \delta = 0$!
- zadatak kaže $k^2 = \lambda^2 \Rightarrow c = -\frac{\beta 2\cancel{k^2}}{\cancel{k^2} 3k^2} = -\frac{2}{3}\frac{\beta}{k^2}$

- iz uvjeta $\delta = 0 \Rightarrow \frac{\beta^2}{9k^4} - \frac{u_T^2}{3} = 0 \Rightarrow u_T^2 = \frac{\beta^2}{3k^4}$

- sada dobivene izraze za u_T i ρ ubacimo npr. u jdbu (1):

$$\Rightarrow (1)/^2 \Rightarrow \left[(\rho - \cancel{\psi_m})k^2 + \beta\right]^2 A^2 = u_T^2 k^4 B^2$$

$$\Rightarrow \left(-\frac{2}{3}\frac{\beta^2}{\cancel{k^2}}\cancel{k^2} + \beta\right)^2 A^2 = \frac{\beta^2}{3\cancel{k^4}}\cancel{k^4} B^2$$

$$\Rightarrow \frac{1}{\cancel{9}_3}\cancel{\beta^2} A^2 = \frac{1}{\cancel{3}}\cancel{\beta^2} B^2 \Rightarrow A^2 = 3B^2 \Rightarrow A = \pm\sqrt{3}\,B$$

$$\Rightarrow \psi_3' = \frac{\pm\sqrt{3}B - B}{\pm\sqrt{3}B + B}\psi_1' = \frac{\pm\sqrt{3} - 1}{\pm\sqrt{3} + 1}\psi_1' \Rightarrow \dots \Rightarrow$$

$$\Rightarrow \boxed{\psi_3' = (2 \mp \sqrt{3})\psi_1'}$$

DZ Za isti slučaj naći ovisnost $\omega_2'(\psi_1') = ?$

[8] Izračunajte totalnu potencijalnu energiju po jedinici površine za neutralnu atmosferu ako su površinski tlak i temperatura redom $p_0 = 10^5$ Pa i $T_0 = 300$ K.

Rj. $E_{pTOT} = E_i + E_p$; E_i... ovisi o temperaturi zraka
E_p... ovisi o težini stupca zraka

- promotrimo stupac zraka od tla do vrha atmosfere (z_A)

$$E_i = c_v \int_0^{z_A} \rho T \, dz \quad ; \quad E_p = \int_0^{z_A} \rho g z \, dz = \int_0^{z_A} p \, dz = R \int_0^{z_A} \rho T \, dz \quad (*)$$

OVU JEDNAKOST TREBA DOKAZATI!

- DOKAZ:

$$E_p = \int_0^{z_A} \rho g \, z \, dz \overset{\text{HIDROSTATIKA}}{=} \int_0^{z_A} \left(-\frac{\partial p}{\partial z}\right) z \, dz = -\int_{p_0}^{0} z \, dp =$$

- parcijalna integracija : $z = u \Rightarrow dz = du$
$dp = dv \Rightarrow p = v$

$$= -\left(\cancel{zp \Big|_{0,p_0}^{z_A,0}}^{\,0} - \int_0^{z_A} p \, dz \right) = \int_0^{z_A} p \, dz \quad \checkmark\checkmark_{OK}$$

$$\Rightarrow E_{pTOT} = c_v \int_0^{z_A} \rho T \, dz + R \int_0^{z_A} \rho T \, dz = c_v \int_0^{z_A} \rho T \, dz + \frac{R}{c_v} c_v \int_0^{z_A} \rho T \, dz =$$

$$= \left(1 + \frac{R}{c_v}\right) \underbrace{c_v \int_0^{z_A} \rho T \, dz}_{E_i} = \frac{c_v + R}{c_v} E_i = \frac{c_p}{c_v} E_i$$

$$\Rightarrow E_{pTOT} = \frac{c_p}{\cancel{c_v}} \cancel{c_v} \int_0^{z_A} \rho T \, dz = c_p \int_0^{z_A} \rho T \, dz \quad (**)$$

- za neutralnu atmosferu vrijedi $\frac{\partial \theta}{\partial z} = 0$; $\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \Big/ \ln \Big/ \frac{\partial}{\partial z}$

$$\Rightarrow \frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p p} \underbrace{\frac{\partial p}{\partial z}}_{=-\rho g} = \frac{1}{T}\frac{\partial T}{\partial z} + \frac{g}{c_p} \underbrace{\frac{\rho R}{p}}_{=\frac{1}{T}} = \frac{1}{T}\left(\underbrace{\frac{\partial T}{\partial z}}_{-\gamma} + \underbrace{\frac{g}{c_p}}_{\delta}\right) \Big/ \theta$$

$$\Rightarrow \frac{\partial \theta}{\partial z} = \frac{\theta}{T}(-\gamma + \delta) = 0 \quad \Rightarrow \gamma = \delta$$

- uzimamo jednostavno linearno opadanje temperature s visinom:
$T(z) = T_0 - \gamma z$

$\Rightarrow T(z) = T_0 - \delta z$

- hidrostatika: $\frac{\partial p}{\partial z} = -\rho g = -\frac{p}{RT} g \Rightarrow \frac{\partial p}{p} = -\frac{g}{RT}\partial z$

$\Rightarrow \frac{\partial p}{p} = -\frac{g}{R}\frac{\partial z}{T_0 - \delta z} \Big/ \int_{p_0,0}^{p,z}$; $x = T_0 - \delta z$; $z = 0 \Rightarrow x = T_0$

$\partial x = -\delta \partial z \Rightarrow \partial z = -\frac{\partial x}{\delta}$

$$\Rightarrow \int_{p_0}^{p} \frac{\partial p}{p} = -\frac{g}{R}\int_{T_0}^{T_0-\delta z} -\frac{\partial x}{\delta x} = \frac{g}{R\delta}\int_{T_0}^{T_0-\delta z}\frac{\partial x}{x}$$

$$\Rightarrow \ln p \Big|_{p_0}^{p} = \frac{g}{R\delta}\ln x \Big|_{T_0}^{T_0-\delta z} \Rightarrow \ln\frac{p(z)}{p_0} = \ln\left(\frac{T_0-\delta z}{T_0}\right)^{\frac{g}{R\delta}}$$

$$\Rightarrow p(z) = p_0\left(\frac{T_0-\delta z}{T_0}\right)^{\frac{g}{R\delta}}$$

- vrh atmosfere: $z = z_A$; $\lim_{z\to\infty} p(z) = 0 \Rightarrow p(z_A) = 0$

$$\Rightarrow p(z_A) = 0 = p_0\left(\frac{T_0-\delta z_A}{T_0}\right)^{\frac{g}{R\delta}} \Rightarrow T_0 - \delta z_A = 0 \Rightarrow z_A = \frac{T_0}{\delta}$$

- sada uzmemo izraz (**): $E_{pTOT} = c_p\int_0^{z_A}\rho T\, dz$, a iz izraza (*) slijedi: $\int_0^{z_A}\rho T\, dz = \frac{1}{R}\int_0^{z_A} p\, dz$

$$\Rightarrow E_{pTOT} = c_p\frac{1}{R}\int_0^{z_A} p\, dz = \frac{c_p}{R}\int_0^{z_A} p_0\left(\frac{T_0-\delta z}{T_0}\right)^{\frac{g}{R}\overset{\frac{1}{\delta}}{\frac{c_p}{g}}} dz$$

$$\Rightarrow E_{pTOT} = \frac{c_p p_0}{R}\int_0^{z_A}\left(1-\frac{\delta}{T_0}z\right)^{\frac{c_p}{R}} dz$$

$y = 1 - \frac{\delta}{T_0}z \Rightarrow dy = -\frac{\delta}{T_0}dz \Rightarrow dz = -\frac{T_0}{\delta}dy$

$z = 0 \Rightarrow y = 1$; $z = z_A \Rightarrow y = 1 - \frac{\delta}{T_0}z_A = 1 - \frac{\delta}{\cancel{T_0}}\frac{\cancel{T_0}}{\delta} = 1 - 1 = 0$

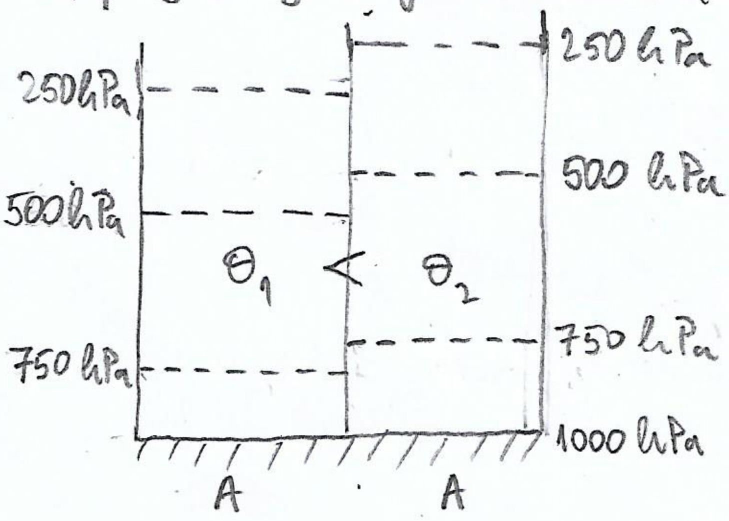
$$\Rightarrow E_{pTOT} = \frac{c_p p_0}{R}\int_1^0 y^{\frac{c_p}{R}}\left(-\frac{T_0}{\delta}\right)dy = \frac{c_p p_0 T_0}{R\delta}\int_0^1 y^{\frac{c_p}{R}}dy$$

$$\Rightarrow E_{pTOT} = \frac{c_p p_0 T_0}{R\delta}\frac{1}{\frac{c_p}{R}+1}y^{\frac{c_p}{R}+1}\Big|_0^1 = \frac{c_p p_0 T_0}{\cancel{R}\frac{g}{c_p}}\frac{\cancel{R}}{c_p+R} = \overbrace{\frac{p_0 T_0}{g}\frac{c_p^2}{c_p+R}}^{2{,}38\cdot 10^9\ \mathrm{Jm^{-2}}}$$

9 Neka su dvije različite zračne mase odvojene zamišljenom pregradom. Svaka od njih ima konstantnu potencijalnu temperaturu $\Theta_1 = 320\,K$ i $\Theta_2 = 340\,K$. Također, svaka zračna masa sjedi na horizontalnoj podlozi površine $A = 1\,ha$ i proteže se od tla do vrha atmosfere. Nađite izraz za dostupnu potencijalnu energiju (APE) ovog sustava.

Rj: APE ćemo dobiti kada oduzmemo totalnu potencijalnu energiju sustava nakon miješanja masa od tot. pot. en. sustava s odvojenim masama: $APE = E_{p1} - E_{p2}$

① prije miješanja zr. masa (sa pregradom) $\Rightarrow$ topliji stupac zraka je viši od hladnijeg



- po jed. površine:

$$E_{pTOT} = \frac{p_0 T_0}{g} \frac{c_p^2}{c_p + R}$$

$$\Theta = T \left(\frac{p_0}{p} \right)^{\frac{R}{c_p}}$$

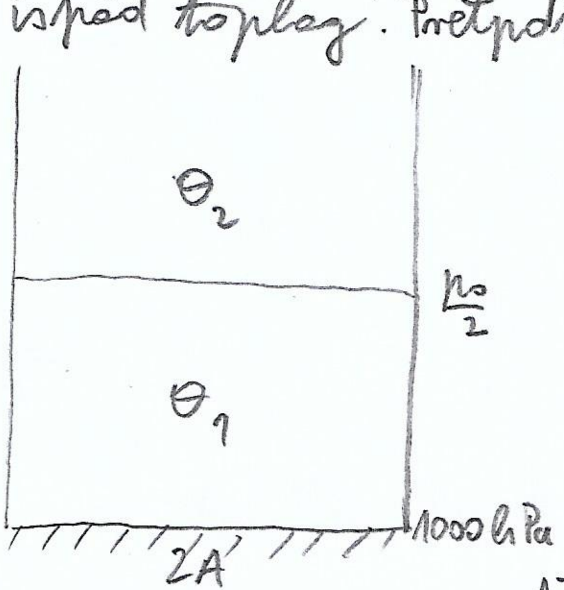
- pri tlu je $T = T_0$ i $p = p_0 \Rightarrow$

$$\Rightarrow \Theta = T_0 \left(\frac{p_0}{p_0} \right)^{\frac{R}{c_p}} = T_0$$

$$\Rightarrow E_{pTOT} = \frac{p_0 \Theta}{g} \frac{c_p^2}{c_p + R} \Rightarrow E_{pTOT\,A} = \frac{p_0 \Theta}{g} \frac{c_p^2}{c_p + R} A$$

$$\Rightarrow E_{p1} = \frac{p_0 \Theta_1}{g} \frac{c_p^2}{c_p + R} A + \frac{p_0 \Theta_2}{g} \frac{c_p^2}{c_p + R} A = \frac{p_0 c_p^2 A}{g(c_p + R)} (\Theta_1 + \Theta_2)$$

② pregrada se "makne" pa će se hladniji zrak podvući ispod toplog. Pretpostavka je da su stupci zraka identične mase pa će se granica uspostaviti točno na $\frac{p_0}{2}$.



- Sada je potrebno izračunati $E_{pTOT\,A}$ novog sustava koji je sada stratificiran!

- općenito: $E_{p\,TOT\,A} = A\frac{c_p}{c_v}E_i = A\frac{c_p}{\cancel{c_v}}\cancel{c_v}\int_0^{z_A} \rho T\, dz =$

$$\rho\, dz = -\frac{dp}{g}$$

$$= -A\frac{c_p}{g}\int_{p_0}^{0} T\, dp$$

$$\Rightarrow E_{p2} = -2A\frac{c_p}{g}\left[\int_{p_0}^{\frac{p_0}{2}} \theta_1\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} dp + \int_{\frac{p_0}{2}}^{0} \theta_2\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} dp\right]$$

$\frac{p}{p_0} = x \Rightarrow dp = p_0\, dx$; $p_0 \to x = 1$; $\frac{p_0}{2} \to x = \frac{1}{2}$; $0 \to 0$

$$\Rightarrow E_{p2} = -2A\frac{c_p}{g}\left[\theta_1 p_0 \int_1^{\frac{1}{2}} x^{\frac{R}{c_p}} dx + \theta_2 p_0 \int_{\frac{1}{2}}^{0} x^{\frac{R}{c_p}} dx\right] =$$

$$= -2A\frac{c_p}{g}\left[\theta_1 p_0 \frac{1}{\frac{R}{c_p}+1} x^{\frac{R}{c_p}+1}\Big/_1^{\frac{1}{2}} + \theta_2 p_0 \frac{1}{\frac{R}{c_p}+1} x^{\frac{R}{c_p}+1}\Big/_{\frac{1}{2}}^{0}\right] =$$

$$= -2A\frac{c_p^2}{g}\frac{p_0}{R+c_p}\left[\theta_1\left(\frac{1}{2}\right)^{\frac{R}{c_p}+1} - \theta_1 - \theta_2\left(\frac{1}{2}\right)^{\frac{R}{c_p}+1}\right]$$

- kada se uvrste brojke, dobije se:

$E_{p1} \approx 5{,}26 \cdot 10^{13}\,J$

$E_{p2} \approx 5{,}23 \cdot 10^{13}\,J \quad \Rightarrow APE \approx 3 \cdot 10^{11}\,J$

- $\overline{DZ}$ $\Rightarrow$ provjeriti brojke!