2010/2011

3. $A : \mathcal{P}_1 \to \mathcal{P}_1$

$$A(p)(t) := t\,p'(t) + p(t)$$

a) $(e) = \{\underset{e_1}{1}, \underset{e_2}{t}\}$

$$A(e_1)(t) = t\,e_1'(t) + e_1(t) = t \cdot 0 + 1 = 1$$

$$A(e_2)(t) = t\,e_2'(t) + e_2(t) = t \cdot 1 + t = 2t$$

$$A(e) = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ A(e_1) \quad A(e_2) \end{array}$$

$$A(e') = \frac{1}{3}\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$$

$$I(e,e') \cdot \Big/ \; A(e') = A(e',e') = I(e',e)\,A(e,e)\,I(e,e') =$$

$$= I(e',e)\,A(e)\,\underbrace{I(e,e')}_{\text{ovo nas zanima}}$$

$$= I(e,e')^{-1} A(e)\, I(e,e')$$

$$\Rightarrow I(e,e') \cdot A(e') = A(e)\, I(e,e') \qquad I(e,e') = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad / \cdot 3$$

$4a+b=3a \Rightarrow a+b=0$

$2a+5b=3b \Rightarrow 2a+2b=0$

$4c+d=6c \Rightarrow d=2c$

$2c+5d=6d \Rightarrow 2c=d$

$\Rightarrow b=-a, \quad d=2c \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -a \\ c & 2c \end{bmatrix}$

$\det I(e,e') = 2ac + ac = 3ac$

$I(e,e')$ mora biti regularna. Možemo uzeti bilo koje

$a, c \in \mathbb{R}\setminus\{0\}$. Npr., $a=c=1$ $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

Provjera:

$$\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} =$$

$$= \frac{1}{3}\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \checkmark$$

$$I(e,e') = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$\quad\quad\quad\quad e_1' \quad e_2'$

$e_1'(t) = 1+t$

$e_2'(t) = -1+2t$

Napomena: Baza (e') nije jedinstvena.

b) $B : \mathcal{P}_3 \to \mathcal{P}_3$

$B(p)(t) := t\,p'(t) + p(t)$

$(e) = \{1, t, t^2, t^3\}$

$B(e_1)(t) = t \cdot 0 + 1 = 1$

$B(e_2)(t) = t + t = 2t$

$B(e_3)(t) = 2t^2 + t^2 = 3t^2$

$B(e_4)(t) = t \cdot 3t^2 + t^3 = 4t^3$

$$\Rightarrow B(e) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Pretpostavimo da postoji baza u kojoj je

$$B(e') = \begin{bmatrix} 1 & -6 & 0 & -1 \\ 2 & 4 & 0 & 0 \\ 0 & 4 & -4 & -3 \\ -1 & 0 & 6 & 3 \end{bmatrix}$$

Tada su $B(e')$ i $B(e)$ slične matrice pa imaju isti trag $\Rightarrow \operatorname{tr} B(e') = \operatorname{tr} B(e)$

$$\Rightarrow \quad 4 = 10 \quad \Rightarrow\Leftarrow$$

Dakle, baza (e') s traženim svojstvima ne postoji.

$\boxed{2011/2012 \quad 2.}$ $A : \mathcal{P}_1 \to \mathcal{P}_1 \qquad (Ap)(t) := p(0) + p(1)t$

$$A(e') = \begin{bmatrix} a & -\frac{1}{2} \\ a & \frac{3}{2} \end{bmatrix}$$

rj: $(e) = \{1, t\}$

$$\left.\begin{aligned} (Ae_1)(t) &= 1 + t \\ (Ae_2)(t) &= 0 + 1 \cdot t = t \end{aligned}\right\} \Rightarrow A(e) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A(e') \sim A(e) \Rightarrow \operatorname{tr} A(e') = \operatorname{tr} A(e)$$

$$\&$$

$$\det A(e') = \det A(e)$$

$$\Rightarrow a + \frac{3}{2} = 2 \Rightarrow a = \frac{1}{2}$$

$$\&$$

$$\frac{3}{2}a + \frac{1}{2}a = 1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$A(e') = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\underbrace{I(e,e') \cdot A(e')}_{A(e,e')} = \underbrace{A(e) \cdot I(e,e')}_{A(e,e')} \qquad I(e,e') = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad / \cdot 2$$

$$\Longrightarrow \begin{aligned} x_1 + x_2 &= 2x_1 \\ -x_1 + 3x_2 &= 2x_2 \\ x_3 + x_4 &= 2x_1 + 2x_3 \\ -x_3 + 3x_4 &= 2x_2 + 2x_4 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_2 &= x_1 \\ x_2 &= x_1 \\ x_4 &= 2x_1 + x_3 \\ x_4 &= 2x_2 + x_3 \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_1 \\ x_3 & 2x_1 + x_3 \end{bmatrix}$$

$$\det \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = x_1 \cdot (2x_1 + x_3) - x_1 x_3 = 2x_1^2$$

$\det \underbrace{I(e,e')}_{\text{regularna}} = 2x_1^2$ Moramo uzeti $x_1 \neq 0$, x_3 može biti bilo koji $\in \mathbb{R}$.

Npr. za $x_1 = 1$, $x_3 = 0$ $\quad I(e,e') = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ \begin{array}{cc} e_1' & e_2' \end{array} \end{array}$

$e_1'(t) = 1$

$e_2'(t) = 1 + 2t$

Jedna takva baza je $(e') = \{1, 1+2t\}$

2013/2014 ②

Dio rješenja vidjeti na webu.

$$A(f)=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$I(e,f)\cdot / \quad A(f)=I(f,e)\cdot A(e)\cdot I(e,f)$$

$$I(e,f)\cdot A(f)=A(e)\cdot I(e,f)$$

$$I(e,f)=\begin{bmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{bmatrix}\cdot\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}0&1&2\\0&0&2\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{bmatrix}$$

$\Rightarrow$ $0 = x_4 + 2x_7$

$x_1 = x_5 + 2x_8$

$x_2 = x_6 + 2x_9$

$0 = 2x_7$

$x_4 = 2x_8$

$x_5 = 2x_9$

$0 = 0$

$\boxed{x_7 = 0}$

$\boxed{x_8 = 0}$

$\Rightarrow$ $x_4 = 0$

$x_1 = x_5$

$x_2 = x_6 + 2x_9$

$x_7 = 0$

$x_4 = 0$

$x_5 = 2x_9 \Rightarrow x_9 = \frac{1}{2}x_5 = \frac{1}{2}x_1$

$$I(e,f) = \begin{bmatrix} x_1 & x_6 + x_1 & x_3 \\ 0 & x_1 & x_6 \\ 0 & 0 & \frac{1}{2}x_1 \end{bmatrix}$$

$$\det I(e,f) = \frac{1}{2}x_1^3$$

$I(e,f)$ regularna $\Rightarrow$ možemo uzeti $x_1 \neq 0$

x_3 i x_6 mogu biti bilo koji.

Stavimo $x_1 = 1$, $x_3 = \frac{1}{2}$ $x_6 = -1$

$$I(e,f) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$(f) = \left\{1, t, \frac{1}{2}(1-2t+t^2)\right\} = \left\{1, t, \frac{1}{2}(t-1)^2\right\}$$

<u>Napomena:</u> Mogli smo uzeti npr. i

$x_1 = 1$, $x_3 = x_6 = 0$.

Tada je
$$I(e,f) = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ \begin{matrix} f_1 & f_2 & f_3 \end{matrix} \end{array}$$

$$\Rightarrow (f) = \left\{1, 1+t, \frac{1}{2}t^2\right\}$$