Gravity and Entropy

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In this paper we investigate the entropy bounds and the properties of geometrical entropy. We derive the Bekenstein and spherical bound. We define and comment on the covariant entropy bound conjecture. Motivated by the hypothesized close relationship between the maximum amount of entropy a region of spacetime can contain and the surface area which in some sense bounds it, we find the properties of the geometrical entropy on (1+1)- and (d+1)- Minkowski spacetime assuming only general properties of quantum entropy.

I. INTRODUCTION

Entropy measures the number of microscopic states which satisfy the macroscopic state of the system, famously given by the Boltzmann formula:

$$S = k_B \ln W \tag{1}$$

where S is the entropy and W is the number of microscopic states. The constant k_B gives the right dimension in thermodynamics, from now on it will be set to $k_B = 1$. In addition, $c = G = 1$, unless explicitly stated otherwise.

In the second part of the 20th century new light has been shed on entropy as the fields of General Relativity and Quantum Field Theory developed. It was recognized that there exists a bound on the amount of entropy a certain amount of matter can contain.

To give a motivation on why that may be true, consider the situation where in the initial state there is a black hole and a certain amount of matter, sufficiently far away from each other. The matter itself has some amount of entropy S_m which obeys the known theormodynamical laws. A stationary black hole is on the other hand characterized by 3 quantities: its mass, angular momentum and charge (this is the famous no-hair theorem[1]). We can imagine the matter falling into the black hole, which eventually settles to a stationary state, which is then unique, as the no-hair theorem tells us. Now, the total entropy is $S_{tot} = 0$, whereas before the matter fell behind the black hole horizon it was $S_m > 0$. That means that second law of thermodynamics, namely $dS_{tot} \geq 0$, has been violated.

The Hawking area theorem [1] states that the area A of black hole is non-decreasing with time: $dA \geq 0$. That reminds us of the second law of thermodynamics. Based on this Bekenstein introduced the black hole entropy: $S_{BH} = kA$, where the constant k was later set to be $\frac{1}{4}$ by Hawking in his black hole temperature calculation[2]. That means that for the entropy of a black hole we have:

$$S_{BH} = \frac{1}{4}A \tag{2}$$

Bekenstein proposed to fix the above mentioned violation of second law by introducing the generalized second law of thermodynamics (GSL), where the total entropy now includes the black hole entropy S_{BH} as well:

$$dS_{tot} = dS_m + dS_{BH} \geq 0 \tag{3}$$

The lesson here is that a stationary black hole indeed has a non-zero entropy, i.e. there are $e^S = e^{A/4}$ microscopic states which satisfy the macroscopic state of the black hole, given by the three variables of the no-hair theorem.

Now, we can come back to the question of existence of matter entropy bound. First entropy bound we will derive is the Bekenstein bound [3].

Suppose there is an amount of matter which has a total energy E and suppose that R is the radius of the largest sphere which contains all of the matter. Furthermore, we will assume that the system is weakly gravitating which is a convenient assumption since it leads to the stability of R in the process of adding the matter to the black hole. Now, to obtain the entropy bound on that matter system we will bring it from infinity to the Schwarzschild black hole which has a radius a, and is much larger than R. This approximation is simply a convenience in the following calculation. Now, we would like the matter to fall into the black hole, but adding the least possible amount of energy to the black hole, which will result in the least possible increase of area of the black hole, and according to (2) the smallest increase in its entropy and thus the best constraint on the entropy of matter system according to GSL (3). We can do this through the so-called Geroch process in which we lower the matter slowly near the horizon. The mass that we add to the black hole will then be the redshifted energy E, where the redshift is calculated at the point of center of mass of the system and the system is sitting just outside of the black horizon, while the sphere of radius R touches the horizon. Thus we need to calculate the redshift factor at the proper distance R from the horizon.

Let x be the coordinate distance from the position outside the black hole (denoted by r) to the horizon:

$$x = r - a$$

The redshift factor of a Schwarzschild black hole at a coordinate position r is given by:

$$\chi(r) = \sqrt{1 - \frac{a}{r}}$$

Now, for $r = a + x$, $x \ll a$:

$$\chi(x) = \sqrt{xa}$$

The proper distance $l(x)$ is related to the coordinate distance x as:

$$l(x) = \int_0^x \frac{dx}{\chi(x)} = 2\sqrt{ax}$$

Hence the redshift factor at the proper distance l is $\chi(l) = \frac{l}{2a}$ and the mass added to the black hole is:

$$\delta M \leq E\chi(l = R) = E\frac{R}{2a}$$

The less-or-equal sign there reminds us of the setup of the process. Recall we supposed that the center of mass is at R which is the extremal scenario: generically we would expect that the matter circumscribed by a sphere of radius R, without any other assumptions, would have a center of mass which is not at R and thus we have the freedom to orient it so it is the closest to the horizon, i.e. at a proper distance less than R which leads to a smaller addition of mass to the black hole. The change in black hole entropy is then:

$$\delta S_{BH} = \frac{dS_{BH}}{dM}\delta M \leq 2\pi ER$$

where we used (2) for S_{BH} and the fact that the surface of Schwarzschild black hole (with Sch. radius R_S) is $A = R_S^2\pi$, and $R_S = 2M$. From the GSL (3) it follows that the matter entropy was at most:

$$S_m \leq 2\pi ER \qquad (4)$$

From the Bekenstein bound (4) we can derive a weaker, spherical bound [**?**][5]. Assume that the matter system is spherically symmetric. Furthermore, assume that it is not a black hole, i.e. it is true for the sphere of radius R which contains the matter of total mass M that $R \geq 2M$ (recall that equal sign is true for the Schwarzschild radius). Then:

$$S_m \leq 2\pi MR \leq 2\pi\frac{R}{2}R = \frac{A}{4}$$

i.e. the matter entropy is less or equal to one quarter of the area of the smallest sphere in which it is contained.

It is tempting to conjecture a generalized bound based on the above derived spherical bound. A quick way to show that we can't postulate that entropy content in a volume V with a boundary of surface size A follows the law:

$$S_m(V) \leq \frac{A}{4}$$

is in the example of Friedmann–Robertson–Walker (FRW) spacetime. In the FRW spacetime the entropy density (entropy in a unit volume) is constant, $\sigma =$ const. That means that the total entropy in a volume V is:

$$S = \sigma V$$

and the proposed bound would be:

$$S \leq \frac{A}{4}$$

However, since the volume grows as $V \propto R^3$ and the area as $A \propto R^2$ it is clear that no matter how small the entropy density is, we can always find such R that the bound is breached.

A need to generalize the Bekenstein bound motivated the so-called covariant entropy bound of Bousso [6]. While the Bekenstein requires the matter system to be gravitationally stable and thus of a stable size in order for the Geroch thought experiment to work, the covariant bound does not.

II. THE COVARIANT ENTROPY BOUND

The covariant entropy bound conjecture states:

Pick any 2-dimensional connected spatial surface B in 4-dimensional spacetime M on which Einstein's equations are satisfied and the matter satisfies the dominant energy condition. Let A be area of the 2-dimensional surface B. Let L be a hypersurface bounded by B and generated by one of the four null congruences orthogonal to B. Additionally, let S be the entropy contained on the hypersurface L. If the expansion of congruence is non-positive at every point on L, then $S \leq A/4$.

In other words, if we pick a 2-dimensional connected spatial surface B, there will be four families of light-rays orthogonal to B. At least 2 of them will have a non-positive expansion. We choose one of them and construct a null hypersurface L by following every light-ray in the chosen family until it either reaches a boundary or a singularity of the space-time or their expansion becomes positive. The conjecture then says that the entropy on a lightsheet L constructed in such a way satisfies the condition that it doesn't exceed one quarter of surface area A of B in natural units:

$$S(L) \leq \frac{A}{4} \qquad (5)$$

Note that this entropy bound has several theorethical advantages over the Bekenstein bound in terms of general applicability: first it is, as the name says, covariant. Secondly, it has a well defined and reasonable condition on the matter, that it needs to satisfy the energy condition. And finally, it does not require a matter system to be gravitationally stable.

The simplest example to explain the choice of light-ray family is when we choose a closed, empty sphere for a 2D surface B. Trivially, there are four light-ray families orthogonal to the surface: 2 of them in the future light cone travel orthogonally to its surface towards the inside or the outside of the sphere (Fig.1.a). The one which travels towards the outside of the sphere has a positive expansion θ. A quick calculation for a little piece of sphere's surface a through which some of the outgoing light rays travel gives that if R is the radius of the sphere, r the standard spherical coordinate and $\Delta\Omega$ a small spatial angle, and λ the affine parameter, then for

$a(R) = R^2 \Delta\Omega$:

$$\theta = \frac{1}{a}\frac{da}{d\lambda} = \frac{1}{R^2\Delta\Omega} 2R\frac{dr}{d\lambda}\Delta\Omega > 0$$

(for the *outgoing* light rays). For the ingoing light rays the $\frac{dr}{d\lambda}$ is negative and thus the expansion θ is negative. The ingoing family is one of the possible choices for the construction of L.

The other two families are in the past light cone of the sphere (Fig.1.b). The same as before, we can calculate the expansions and we would see that the outgoing family has $\theta > 0$ and the ingoing has $\theta < 0$, which means the latter would be another good choice for the purpose of the conjecture.

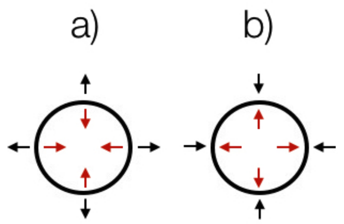

FIG. 1: a) Light rays in the future cone, travelling towards the inside (red) and the outside (black) of the sphere. b) Light rays in the past cone, incoming from the inside (red) and the outside (black) of the sphere.

It is now convenient to turn our attention to the fact that we require the dominant energy condition for the matter to hold. The *dominant energy condition*[1] says that for every causal (lightlike or timelike) four-vector V^a:

$$-T^a_b V^b = W^a$$

the stress-energy tensor T_{ab} satisfies the condition if W^a is also a causal four-vector. It is equivalent to the statment that nothing can travel faster than light.

It is important in the context of understanding where do we stop following a light-ray in the construction of a light sheet. That happens when the expansion changes the sign from a negative to a positive one. The Raychauduri equation for the change of expansion θ with change of affine parameter λ for a congruence of null geodesics with tangent vector field k^a is as follows:

$$\frac{d\theta}{d\lambda} = \frac{1}{2}\theta^2 - 8\pi\, T_{ab}k^a k^b - \sigma^2 + \omega^2 \tag{6}$$

where T_{ab} is the stress-energy tensor of matter, $\sigma^2 = \sigma_{ab}\sigma^{ab}$ and $\omega^2 = \omega_{ab}\omega^{ab}$ are non-negative quadratic invariants of the shear and vorticity tensors. Vorticity tensor is in the case of surface-orthogonal null congruence equal to zero. The dominant energy condition implies the null energy condition: $T_{ab}k^a k^b \geq 0$. Thus we conclude that the right-hand side of eq. (6) is non-positive and that expansion does not increase along any geodesic. Since:

$$\frac{d\theta}{d\lambda} \leq \frac{1}{2}\theta^2$$

a simple integration will gives us

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta_0} + \frac{1}{2}\lambda.$$

Thus, if the expansion is a negative θ_0 at any point of a light ray (null geodesic) in the congruence, it will after a finite affine time

$$\Delta\lambda \leq \frac{2}{|\theta_0|}.$$

diverge to $\theta \to -\infty$.

At this point the nearby light rays are reaching a single focus point, after which they start to expand. This is called a *caustic*. Since at the focus point the expansion changes sign from negative to positive, it is the point where we stop following the light-ray and these are the endpoints of the light-sheet.

Now, back to the simple example. We understood how to choose a light-ray family, and now we turn attention to the construction of lighsheet L. The two ingoing families are the good choice. They both end in the center of the sphere, at the caustic. Clearly, at that point the expansion changes the sign from negative to positive, which according to the conjecture is the point at which the construction of lighsheet is finished.

To summarize, we choose light-rays from one of the two ingoing families which start at the sphere's surface B (according to the condition in the conjecture requiring the lightsheet to be bounded by B), follow them until the caustic point and call the hypersurface created in that way a lightsheet L. Then we calculate the entropy on L and check if the entropy of matter on L satisfies the conjecture.

While there are a number of proofs of this conjecture (or its slightly generalized version), under special conditions, such as the quantum proof for free fields in the limit where the lightsheet size is small compared to curvature invariants [7], and the case where entropy can be described by an entropy current under certain restrictions [8], the full proof is still out of reach.

It is, however, possible to investigate the newly discovered connection between the limit on the amount of an entropy (or information) in region of spacetime and it's lower-dimensional boundary. Note that the bound on the information content which can be stored in a region of space may be interesting also for practical purposes in the distant future.

III. QUANTUM ENTROPY

Quantum mechanically a physical state is described by the density matrix ρ. The entropy of the quantum state is given by the expression:

$$S = -\text{Tr}(\rho \log \rho) \quad (7)$$

For a pure state $\rho = |\psi\rangle\langle\psi|$ the entropy is equal to zero and it is otherwise always positive. For a maximally entangled system the entropy measures the dimension of Hilbert space d, $S = \log d$, and that is the maximum possible value of entropy. Note that for an infinite-dimensional Hilbert space entropy can be infinite.

If the Hilbert space is a tensor product of two or more spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and ρ is the density matrix on $\mathcal{H}$, one can obtain the density matrix on $\mathcal{H}_1$ as $\rho_1 = \text{Tr}_2\rho$, i.e. tracing the total density matrix on $\mathcal{H}_2$.

For a pure density matrix ρ:

$$S(\rho_1) = S(\rho_2) \quad (8)$$

dute to the fact that $\rho = |\psi\rangle\langle\psi|$, and by Schmidt decomposition we can write the state $|\psi\rangle$ as:

$$|\psi\rangle = \sum_i c_i |\psi_{1,i}\rangle|\psi_{2,i}\rangle,$$

where $|\psi_{i,j}\rangle$ are a set of orthogonal vectors on each of the subspaces $i = 1, 2$. If we take a partial trace of ρ, $\rho_i = \text{Tr}_{j\neq i}\rho$, we get:

$$\rho_i = \sum_k \left|c_k^2\right| |\psi_{i,k}\rangle\langle\psi_{i,k}| \, .$$

It is clear that the reduced density matrices have the same eigenvalues, which implies (8).

Interestingly, if ρ is a pure state with $S(\rho) = 0$, generically ρ_1 and ρ_2 will be in a mixed state with $S(\rho_i) > 0$. Take for example $\rho = |\psi\rangle\langle\psi|$ where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2$$

is a Bell state and

$$\rho_1 = \frac{1}{2}(|0\rangle_1\langle 0|_1 + |1\rangle_1\langle 1|_1).$$

The entropy of the reduced matrix is $S(\rho_1) = log(2)$, the same as for ρ_2. This tells us that, *generically*, the entropy does not increase with the size of the system.

If the density matrix describes two independent systems, $\rho = \rho_1 \otimes \rho_2$, it is easily seen form eq. 7 that $S(\rho) = S(\rho_1) + S(\rho_2)$. In general:

$$S(\rho) \leq S(\rho_1) + S(\rho_2) \quad (9)$$

We will generalize the above inequality for a physical system with a Hilbert space which is a tensor product of an arbitrary number of subspaces, $\mathcal{H} = \otimes_{i\in I}\mathcal{H}_i$, where I is a set of indices $I = 1, 2, ...$ labelling the different subspaces. We will define the reduce density matrix ρ_A on $\otimes_{i\in A}$, where $A \in I$ by tracing over all subspaces $\mathcal{H}_{i\notin A}$:

$$\rho_A = \text{Tr}_{\otimes_{i\notin A}\mathcal{H}_i}\rho$$

$S(A) := S(\rho_a)$ is the entropy of such a subsystem. Now, according to (9) for any subsets A and B of I, it is true that:

$$S(A) + S(B) \geq S(A \cup B) \quad (10)$$

By (8) we have $S(A) = S(-A)$ where $-A$ is the set complementary to A in I. Now we have $S(-A) + S(-B) \geq S(A \cup B) = S(-(A \cap B))$, where the complement operation changed union into an intersection. Finally we use (8) to arrive at:

$$S(A) + S(B) \geq S(A \cap B) \quad (11)$$

Strong subadditivity (SSA)[9] is a property of quantum entropy which says that for Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and the corresponding density matrices: ρ on $\mathcal{H}$, $\rho_2 = \text{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_3}\rho$ and $\rho_{12} = \text{Tr}_{\mathcal{H}_3}\rho$ (and others analogously defined) it is true that:

$$S(\rho_{12}) + S(\rho_{23}) \geq S(\rho) + S(\rho_2) \quad (12)$$

SSA (12) generalizes (10) and (11) to:

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B) \quad (13)$$

From that we can derive using the complementary entropies equality (8) and the trick with changing union to intersection with complementarity operation the *SSB inequality* (named for brevity and close relation to SSA):

$$S(A) + S(B) \geq S(A - B) + S(B - A) \quad (14)$$

where $A - B = A \cap (-B)$. Finally, for non-intersecting A and B SSA and SSB imply:

$$|S(A) - S(B)| \leq S(A \cup B) \leq S(A) + S(B) \quad (15)$$

IV. PROPERTIES OF THE GEOMETRIC ENTROPY

Now that we have all the necessary inequalities we will derive geometrical properties of entropy. So far we have used general notions and results in quantum theory. Let us turn our attention to continuous quantum systems in $\mathbb{R}^d$. The Hilbert space will be the Fock space, $\mathcal{H} = \oplus_0^\infty \mathcal{H}_n$, where the n-particle space $\mathcal{H}_n$ is the tensor product of n copies of $\mathcal{H}_1$, and $\mathcal{H}_1 = \mathcal{L}^2(\mathbb{R})$ is the Hilbert space of sqaure integrable functions in $\mathbb{R}$. We can also define the Fock space on a subset of $\mathbb{R}^d$, volume $V \in \mathbb{R}^d$. The one-particle space is now the space of square integrable functions on V and total space is simply analogously $\mathcal{H}(V) = \oplus_0^\infty \mathcal{H}_n(V)$.

If we have two volumes V_1 and V_2 which do not intersect (or have a zero measure intersection) then we have

$$\mathcal{H}(V_1 \cup V_2) = \mathcal{H}(V_1) \otimes \mathcal{H}(V_2). \tag{16}$$

That is equivalent to the statement that the two systems are independent, which is clear from the fact that the two volumes do not intersect and thus the one-particle states bounded by the respective volumes are independent of each other, which implies the independence of n-particle states on V_1 and V_2 for any n.

Now we can take that the density matrix of a system to be ρ. ρ_V will be the reduced density matrix on V, which contains all of the information of the physical system restricted to the volume V and may be computed from ρ as the trace over complement of V in the entirety of the volume of the system:

$$\rho_V = \mathrm{Tr}_{\mathcal{H}} \rho_{V \cup V'} \tag{17}$$

where V' is any volume which has measure zero intersection with V. The entropy of ρ_V is

$$S(\rho_V) \equiv S(V) = -\mathrm{Tr} \rho_V \log \rho_V .$$

Now we have specified that we are dealing with a continuous quantum systems as defined above, and we can apply results derived in the previous section.

We will additionally use the property that when the total state describe by ρ is invariant under some transformation U, then the entropy $S(U(A)) = S(A)$, where A is a subset of the total space of the system.

We will show properties of entropy on (1+1)- and (d+1)-Minkowski spacetime. A few notes are in order.

We will use Penrose diagrams of the Minkowski spacetime (Fig. 2) to depict the different situations in the following paragraphs.

A Cauchy surface of a spacetime is any subset which is intersected by a causal, inextensible curve exactly once. Informally, the initial conditions on a Cauchy surface of a spacetime, uniquely determines the past and future. In the special case of Minkowski space, for example, any spatial hypersurface of constant time t is a Cauchy surface.

We assign a density matrix with respective entropy to any subset A of a Cauchy surface $\mathcal{C}$ and assume the conditions (16) and (17) for them to hold in order to be able to use results from sec. 3.

Consider the case of subsets A of a Cauchy surface $\mathcal{C}$ and A' of a Cauchy surface $\mathcal{C}'$ which have the same causal development $\mathcal{A}$ (i.e. they both uniquely determine the same $\mathcal{A}$ which is a subset of Minkowski space) (Fig. 3). The unitarity of a causal evolution implies that

$$S(A) = S(A') \equiv S(\mathcal{A}) \tag{18}$$

This is because the evolution of a density matrix is unitary, $U^+U = UU^+ = 1$, $\rho(t) = U^+ \rho \, U$ which by the property of trace operation $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ leaves the entropy (7) unchanged.

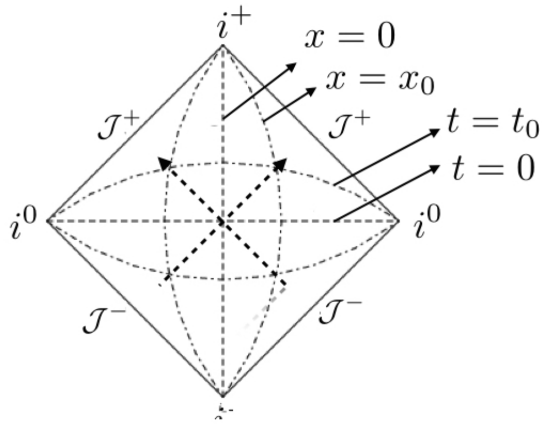

FIG. 2: A Penrose diagram of the (1+1)-Minkowski spacetime. A pair of curves of constant x and t are labeled. $\mathcal{J}^+$ and $\mathcal{J}^-$ are respectively the null future and past infinities. ß$_0$ is the spacelike infinity and $i^\pm$ are the timelike future and past infinities. The direction of photon rays is shown by dashed arrows in the center of the diagram. Photons always travel at 45^o incline from the vertical axis from past null infinity towards future null infinity.

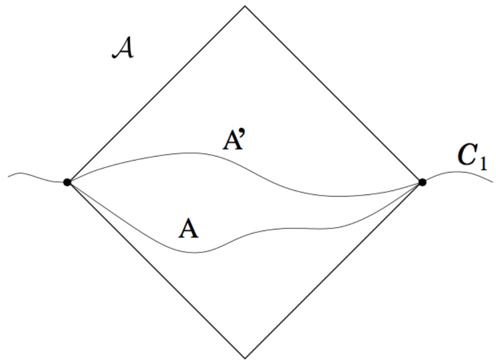

FIG. 3: [10] $\mathcal{A}$ is a *causally closed* subset. All the causal trajectories which pass through its Cauchy surface are contained in it. Two different subsets A of a Cauchy surface $\mathcal{C}$ and A' of a Cauchy surface $\mathcal{C}'$ which have the same causal development $\mathcal{A}$.

We will say that two causally closed sets $\mathcal{A}$ and $\mathcal{B}$ commute if there is at least a pair of the respective Cauchy surface representatives that belong to the same global Cauchy surface. This is equivalent to their spatial corners (i^0) being spacelike separated, as explained in Fig. 4.

The notion of commuting sets is important due to the fact that the SSA and SSB inequalities can only be applied to such sets. Recall that we need the properties (16) and (17) to be well defined. While this requires sets A and B to be subsets

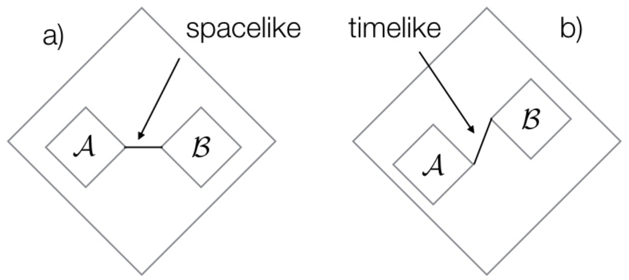

FIG. 4: Two causally closed sets $\mathcal{A}$ and $\mathcal{B}$, subsets of a Minkowski spacetime which a) commute (spacelike separation of corners), b) do not commute (time- or null-like separation of corners) as defined in the text.

of the same global Cauchy surface $\mathcal{C}$ and, moreover spatial and spatially separated from each other, we can by property (18) extend it to A and B being any Cauchy surface of $\mathcal{A}$ and $\mathcal{B}$, respectively, if $\mathcal{A}$ and $\mathcal{B}$ commute.

Theorem 1 The most general form for a relativistic entropy function on a subset Z in (1+1)-dimensional Minkowski space is given by

$$S(Z) \equiv S_m = \gamma + (m-1)\beta \tag{19}$$

where m is the number of connected components of Z, γ is defined as the entropy on a vanishingly small Cauchy surface of size x

$$\gamma = \lim_{x\to 0} S(x)$$

and $\gamma \geq \beta \geq 0$.

We will prove this theorem by induction. First we need to show that it's true for $m = 1$ and $m = 2$.

First we note that a connected causally closed set is defined by the size of the uniquely defined Cauchy surface, which is the size of the base of the Penrose diamond as shown in fig. 5.

Next we define (the same as in the theorem statement)

$$\gamma = \lim_{x\to 0} S(x).$$

The entropy $S(x)$ is a positive, non-decreasing and a concave function [10].

The boost symmetry of Minkowski spacetime gives an additional constraint to the entropy. If we apply SSA (13) to the construction in fig. 6, we have

$$S(x) + S(x') \leq 2S(\sqrt{xx'}) \tag{20}$$

We can take the limit of (20) when $x' \to 0$ and obtain $S(x) \leq \gamma$. Now, since $S(x)$ is non decreasing and $\gamma = \lim_{x\to 0} S(x)$, we find that

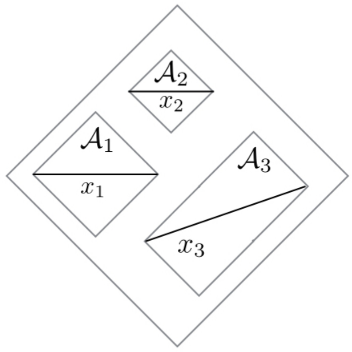

FIG. 5: The uniquely defined Cauchy surfaces for connected causally closed sets $\mathcal{A}_i$ of size x_i

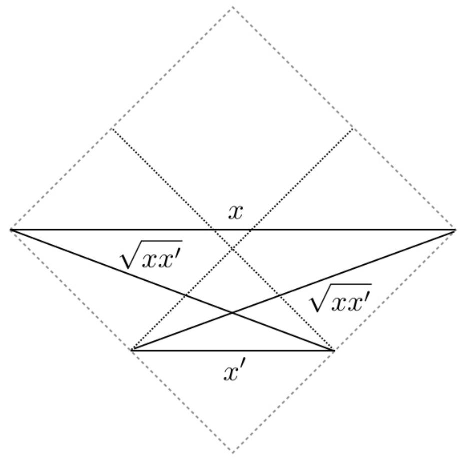

FIG. 6: The diamond with the Cauchy surface size x' is the intersection of two the diamonds which commute (their spatial corners are spatially separated) and have a Cauchy surface size $\sqrt{xx'}$. The diamond with the Cauchy surface size x is their union with causal completion. We perform

$$S(x) = \gamma = \text{const.} \tag{21}$$

Notice that this proves validity of the theorem for $m = 1$, since it is the entropy of a connected causally closed set.

Now we turn to 2-component sets. In figure 7 the set A has 2 disconnected components. The set B commutes with A so we can use the SSA inequality. We define a new 2-component set A' as the union $A' = A \cup B$.

We have

$$S(A) + S(B) = S(A) + \gamma$$

where we used the previous result for the entropy of a connected set for B. Combined with $S(A \cap B) = S(A')$ and $S(A \cup B) = \gamma$, and applying the SSA inequality we get

$$S(A) + \gamma \geq S(A') + \gamma \tag{22}$$

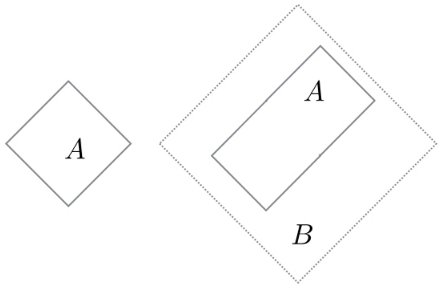

FIG. 7: Two component causally closed set A. B is a connected causally closed setwhich contains one part of A.

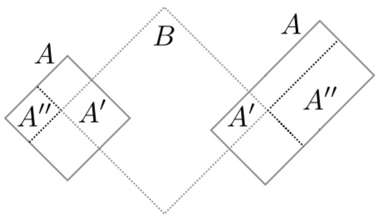

FIG. 8: Two component set A .A' components share its the inner edges with A, while A'' components share its the outer edges with A. Clearly B and A commute with each other due to the fact that there is a pair of the respective Cauchy surface representatives that belong to the same global Cauchy surface.

In the figure 8 we have a different construction. A' and A'' are included in A. A and B commute. By applying the SSA inequality we get

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) = \gamma + S(A')$$

$$S(A) + \gamma \geq S(A') + \gamma \tag{23}$$

Now we see from (22) that for any 2-component sets A' and A, where $A \in A'$, $S(A') \leq S(A)$. On the other hand, from (23) we see that for any $A \in A'$, $S(A') \geq S(A)$. This set of inequalities implies that the entropy of a 2-component set hast to be a constant

$$S(A) = \text{const.} = \gamma + \beta$$

where we defined the constant β so that the form of entropy would match the one given in theorem 1.

We have proved the theorem 1 for the case $m = 1$ and $m = 2$. Now we turn to induction. Assume the theorem is correct for every $n' \leq n$.

Now from the figures 9a) and 9b) we derive analogously to the previous inequalities

$$S_{n+1} + \gamma \geq S_n + \delta$$

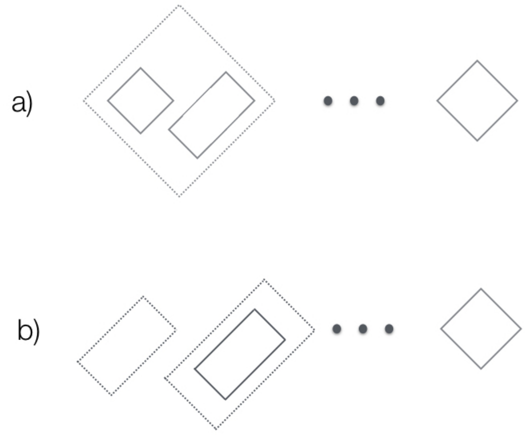

FIG. 9: Construction accompanying the proof by induction.

$$S_n + \delta \geq S_{n+1} + \gamma.$$

These inequalities imply

$$S_{n+1} + \gamma = S_n + \delta$$

which proves the formula in theorem 1. Additionaly, from positivity of entropy $\gamma, \delta \geq 0$. From subadditivity $S_{n+1} \geq S_n$ and thus $\gamma \leq \delta$. That finishes the proof of theorem 1.

This theorem for (1+1)-dimensional Minkowski spacetime agrees with our initial motivation to investigate the relationship between the (maximal) entropy and the size of its boundary surface. The boundary of a finite line are two points. We would have thus initially guessed that the entropy of a system of spatial dimension 1 has a constant entropy which is exactly what we derived.

By analogous procedure the following theorem can be proved in the higher-dimensional Minkowski spacetime:

Theorem 2 The most general form for a relativistic entropy function on the relativistic polyhedra on (d + 1)-dimensional Minkowski spacetime, with $d > 1$, is given by

$$S(X) = \alpha_0 + s\ \text{area}(X), \tag{24}$$

where α_0 and s are non-negative constants and area(X) is the area of the boundary of the Cauchy surface of X.

The subtleties in the proof of this theorem lie in the fact that the commutation property for 2 polyhedra in higher dimensions are more complicated, however, the arguments remain essentially the same.

The interesting result of this theorem is that we found a direct relationship between entropy of a subset of the Minkowski spacetime and the size of the boundary of its Cauchy surface.

V. CONCLUSION

The Bousso covariant bound on entropy hypothesizes that the amount of entropy or information that we can store in a region of spacetime is bounded by surface size of its boundary.

There are strong evidences hinting at the validity of this conjecture, which is a peculiar thing from the stand point of the theories we have at the moment, which are local. Quantum field theory has degrees of freedom at every point in space. We would expect then that the information content of should grow with volume of a spatial region.

We investigated how close is the connection between entropy and surface size. We used only the general properties of the quantum entropy.

The symmetries of the Minkowski spacetime and the strong subadditivity property of quantum entropy were shown to be surprisingly powerful tools. We found that the entropy function on the (1+1)-dim. Minkowski space has two constant terms, one of which is proportional to the number of connected components. On (d+1)-dim space we find that there is again a constant contribution to the entropy and, more importantly a term which is proportional to the area of the boundary of Cauchy surface of a polyhedral subset of space.

The entropy of which we derived the properties is called geometric entropy. We suppose in the calculation that we may take a subset of a spacetime and trace the vacuum state over the outside of the subset and obtain a density matrix and the corresponding quantum entropy. That is not necessarily a valid operation, since in QFT calculations divergences appear when we do so.

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