Rješenja 1. kolokvija

1. a) $S : \mathbb{R}^4 \to \mathbb{R}^3$

$$S(x,y,z,w) = (x-y+w,\ 2y-z,\ 2x-z-4w)$$

Prvo ćemo naći $\mathrm{Ker}\, S$:

$$S(x,y,z,w) = (0,0,0) \iff \begin{array}{c} x-y+w=0 \\ \& \\ 2y-z=0 \\ \& \\ 2x-z-4w=0 \end{array}$$

$$\iff z=2y \ \&\ w=y-x \ \&\ 2x-2y-4(y-x)=0$$

$$\iff z=2y \ \&\ w=y-x \ \&\ 6x-6y=0$$

$$\iff y=x \ \&\ z=2x \ \&\ w=0$$

$$\Rightarrow \mathrm{Ker}\, S = \{(x,x,2x,0) \mid x\in\mathbb{R}\} =$$

$$= [\{(1,1,2,0)\}] \qquad \text{(2 boda)}$$

Jedna baza za $\mathrm{Ker}\, S$ je $\{(1,1,2,0)\}$

$$\Rightarrow d(S) = \dim \mathrm{Ker}\, S = 1.$$

Po teoremu o rangu i defektu,
$r(S) + d(S) = \dim \mathbb{R}^4 = 4$. Stoga je
$r(S) = 4 - d(S) = 4 - 1 = 3$. (1 bod)

$$\left.\begin{array}{r} \Rightarrow \dim \operatorname{Im} S = \dim \mathbb{R}^3 \\ \operatorname{Im} S \leq \mathbb{R}^3 \end{array}\right\} \Rightarrow \boxed{\operatorname{Im} S = \mathbb{R}^3}$$

Jedna baza za $\operatorname{Im} S$ je $\{(1,0,0), (0,1,0), (0,0,1)\}$.
(1 bod)

b) Pretpostavimo da postoje $a, b, c \in \mathbb{R}$ takvi da je T monomorfizam. Tada je $d(T) = 0$. Po teoremu o rangu i defektu, $r(T) + d(T) = \dim \mathbb{R}^4 = 4$.

$$\left.\begin{array}{r} \Rightarrow r(T) = 4 \\ \operatorname{Im} T \leq \mathbb{R}^3 \end{array}\right\} \Rightarrow \Leftarrow$$

Dakle, ne postoje $a, b, c \in \mathbb{R}$ t.d. je T monomorfizam. (1 bod)

② a) $f^*: \mathcal{P}_3 \longrightarrow \mathbb{R}$

$f^*(p) = p(1) - p'(1)$

$$B = \{\underset{\substack{\| \\ e_1}}{-1}, \underset{\substack{\| \\ e_2}}{1+t}, \underbrace{-1-t-t^2}_{e_3}, \underbrace{1+t+t^2+t^3}_{e_4}\}$$

Neka je $\{f_1, f_2, f_3, f_4\}$ dualna baza baze B. Po definiciji dualne baze, vrijedi $f_i(e_j) = \delta_{ij}$.

Neka je $f^* = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$. Trebamo odrediti koeficijente $\alpha, \beta, \gamma, \delta$.

(1 bod)

$$f^*(e_1) = \alpha f_1(e_1) + \beta f_2(e_1) + \gamma f_3(e_1) + \delta f_4(e_1) =$$
$$= \alpha \cdot 1 + \beta \cdot 0 + \gamma \cdot 0 + \delta \cdot 0 = \alpha.$$

Analogno dobijemo:

$f^*(e_2) = \beta$

$f^*(e_3) = \gamma$

$f^*(e_4) = \delta$.

Stoga je
$$\alpha = f^*(e_1) = e_1(1) - e_1'(1) = -1 - 0 = -1$$
$$\beta = f^*(e_2) = e_2(1) - e_2'(1) = 2 - 1 = 1$$
$$\gamma = f^*(e_3) = e_3(1) - e_3'(1) = -3 + 3 = 0$$
$$\delta = f^*(e_4) = e_4(1) - e_4'(1) = 4 - 6 = -2$$

$$\Longrightarrow \boxed{f^* = -f_1 + f_2 - 2f_4}$$ (1 bod)

b) Zapišimo vektore kanonske baze $\{1, t, t^2, t^3\}$ kao linearnu kombinaciju vektora baze B.

$$
\left.\begin{aligned}
1 &= -(-1) = -e_1 \\
t &= (-1) + (1+t) = e_1 + e_2 \\
t^2 &= -(1+t) - (-1-t-t^2) = -e_2 - e_3 \\
t^3 &= (-1-t-t^2) + (1+t+t^2+t^3) = e_3 + e_4
\end{aligned}\right\} \text{(1 bod)}
$$

$$
\text{(1 bod)} \left\{\begin{aligned}
f_i(a+bt+ct^2+dt^3) &= a\,f_i(1) + b\,f_i(t) + c\,f_i(t^2) + d\,f_i(t^3) \\
&= a\,f_i(-e_1) + b\,f_i(e_1+e_2) + c\,f_i(-e_2-e_3) \\
&\quad + d\,f_i(e_3+e_4), \quad i \in \{1,2,3,4\}
\end{aligned}\right.
$$

$$
\left.\begin{aligned}
f_1(a+bt+ct^2+dt^3) &= -a + b \\
f_2(a+bt+ct^2+dt^3) &= b - c \\
f_3(a+bt+ct^2+dt^3) &= -c + d \\
f_4(a+bt+ct^2+dt^3) &= d
\end{aligned}\right\} \text{(1 bod)}
$$

③ (a) $A(a+bt+ct^2) = a+(a+2b)t+(a+b+c)t^2$

$$(e') = \{\underbrace{1-t}_{e_1'}, \underbrace{1+t^2}_{e_2'}, \underbrace{t^2}_{e_3'}\}$$

$$A(e_1') = 1+(1-2)t+(1-1+0)t^2 = 1-t = e_1'$$

$$A(e_2') = 1+t+2t^2 = -1+t+2(1+t^2) =$$
$$= -e_1' + 2e_2'$$

$$A(e_3') = t^2 = e_3'$$

$$\Rightarrow A(e') = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} A(e_1') & A(e_2') & A(e_3') \end{array} \end{array}$$

(1 bod)

(b) Iz a) dijela zadatka vidimo da je

$$k_A(\lambda) = \det(A(e') - \lambda I) = (1-\lambda)(2-\lambda)(1-\lambda) =$$
$$= (\lambda-1)^2(2-\lambda)$$

$$\Rightarrow \sigma(A) = \{1, 2\}$$

$a(1) = 2, \quad a(2) = 1$ (1)

Znamo $1 \leq g(2) \leq a(2) = 1 \quad \Rightarrow g(2) = 1.$

(1 bod)

Odredimo $g(1)$.

$$A(e) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$A(e) \downarrow$ kanonska baza

(1 bod)

$V_A(1) = Ker(A - I)$ Rješavamo homogeni sustav

$$(A(e) - I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A(e) - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ /\cdot(-1) \\ \swarrow + \end{matrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow V_A(1) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 + x_2 = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ -x_1 \\ x_3 \end{bmatrix} : x_1, x_3 \in \mathbb{R} \right\}$$

$$= \left[\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right]$$

$$\Longrightarrow g(1) = \dim V_A(1) = 2 = a(1) \qquad (2)$$

Iz (1) & (2) $\Rightarrow$ A se može dijagonalizirati.

Uvedimo oznake $f_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Odredimo sada $V_A(2)$.

$V_A(2) = \mathrm{Ker}\,(A-2I)$

$$A(e)-2I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{matrix} \oplus \\ /\cdot(-1) \\ \oplus \end{matrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$V_A(2) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 0 \ \& \ x_2 - x_3 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix} : x_2 \in \mathbb{R} \right\} = \left[\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right]$$

Označimo $f_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$\{f_1, f_2\}$ je baza za $V_A(1)$, a $\{f_3\}$ za $V_A(2)$.

Stoga je $\{f_1, f_2, f_3\}$ baza za $\mathbb{R}^3$.

$$A(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(1 bod)

Stavimo li npr. $(f') = \{f_1, f_1 + f_2, \frac{1}{2} f_3\}$ imamo

$A(f') = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Dakle, baza u kojoj A ima dijagonalni prikaz nije jedinstvena.

(1 bod)

4.

$A : \mathcal{P}_2 \to \mathcal{P}_2$

$(A(p))(t) = p'(2t+1)$

$(e) = \{1, t, t^2\}$

$(A(e_1))(t) = 0$

$(A(e_2))(t) = 1 = e_1(t)$

$(A(e_3))(t) = 2\cdot(2t+1) = 4t+2 = 2e_1(t) + 4e_2(t)$

$$\Rightarrow A(e) = \begin{array}{ccc} & & \\ \left[\begin{array}{ccc} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{array}\right] & & \\ \uparrow \; A(e_1) \quad \uparrow \; A(e_2) \quad \uparrow \; A(e_3) & & \end{array}$$

(1 bod)

Ako postoji baza (f) za $\mathcal{P}_2$ t.d. je

$$A(f) = \begin{bmatrix} 1 & 3 & a \\ -1 & b & 2 \\ 1 & -1 & -2 \end{bmatrix},$$

onda $k_{A(e)}(\lambda) = k_{A(f)}(\lambda)$, $\operatorname{tr} A(e) = \operatorname{tr} A(f)$,

$\det A(e) = \det A(f)$. (jer su $A(e)$ i $A(f)$ slične).

$\boxed{\operatorname{tr} A(e) = \operatorname{tr} A(f) \Rightarrow 0 = b - 1 \Rightarrow \boxed{b = 1}}$ (1 bod)

$\det A(e) = \det A(f) \Rightarrow 0 = 0$

$$k_{A(e)}(\lambda) = k_{A(f)}(\lambda) \Rightarrow \begin{vmatrix} -\lambda & 1 & 2 \\ 0 & -\lambda & 4 \\ 0 & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 3 & a \\ -1 & 1-\lambda & 2 \\ 1 & -1 & -2-\lambda \end{vmatrix}$$

$$\Rightarrow \quad (-\lambda)^3 = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 3 & a \\ -1 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 3 & a \\ 1-\lambda & 2 \end{vmatrix}$$

$$= (1-\lambda)\left((1-\lambda)(-2-\lambda)+2\right) + 3(-2-\lambda)+a+6-a(1-\lambda)$$

$$= (1-\lambda)(\lambda^2+\lambda) - \cancel{6} - 3\lambda + \cancel{a} + \cancel{6} - \cancel{a} + a\lambda$$

$$= \lambda\left((1-\lambda)(1+\lambda)+a-3\right)$$

$$= \lambda(-\lambda^2+a-2) \quad \forall \lambda \in \mathbb{R}$$

$$\Rightarrow \quad -\lambda^2 = -\lambda^2 + a - 2 \quad \forall \lambda \in \mathbb{R}$$

$$\Rightarrow \quad \boxed{a=2} \qquad (1 \text{ bod})$$

Uočimo da su $a=2$, $b=1$ nužni, ali NE i dovoljni uvjeti. Naime, moramo provjeriti postoji li barem jedna baza (f) za $\mathcal{P}_2$ za koju je $A(f) = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$.

Drugim riječima, moramo provjeriti jesu li matrice $A(e)$ i $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$ slične.

Ako postoji baza (f) za $\mathcal{P}_2$ t.d. $A(f) = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$, onda je $A(f) = I(f,e)A(e)I(e,f)$

$\Rightarrow I(e,f)\cdot A(f) = A(e)\cdot I(e,f)$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}}_{I(e,f)} \cdot \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

(1 bod)

Uočite da je dovoljno naći jedno rješenje ovog sustava 9×9 za koje je matrica $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$ regularna. (Ako bi se desilo da takvo rješenje ne bi postojalo, onda bi odgovor bio: „Ne postoje $a,b\in\mathbb{R}$ za koje je matrica $\begin{bmatrix} 1 & 3 & a \\ -1 & b & 2 \\ 1 & -1 & -2 \end{bmatrix}$ matrični prikaz operatora A u nekoj bazi")

$x_1 - x_2 + x_3 = x_4 + 2x_7$

$3x_1 + x_2 - x_3 = x_5 + 2x_8$

$2x_1 + 2x_2 - 2x_3 = x_6 + 2x_9 \quad /:2$

$x_4 - x_5 + x_6 = 4x_7$

$3x_4 + x_5 - x_6 = 4x_8$

$2x_4 + 2x_5 - 2x_6 = 4x_9 \quad /:2$

$$x_7 - x_8 + x_9 = 0$$
$$3x_7 + x_8 - x_9 = 0$$
$$2x_7 + 2x_8 - 2x_9 = 0 \quad / : 2$$

$\oplus \Rightarrow \boxed{x_7 = 0}$

$\Rightarrow \boxed{x_8 = x_9 = t}$

$$x_4 - x_5 + x_6 = 0$$
$$3x_4 + x_5 - x_6 = 4t$$
$$x_4 + x_5 - x_6 = 2t$$

$\oplus \Rightarrow \boxed{x_4 = t}$

$\boxed{x_6 - x_5 = -t}$

$\boxed{x_5 = s}$

$$x_1 - x_2 + x_3 = t$$
$$3x_1 + x_2 - x_3 = s + 2t$$
$$x_1 + x_2 - x_3 = t + \frac{s-t}{2}$$

Stavimo npr. $t = 1$, $s = 1$

$$x_1 - x_2 + x_3 = 1$$
$$3x_1 + x_2 - x_3 = 3$$
$$x_1 + x_2 - x_3 = 1$$

$\oplus \Rightarrow \boxed{x_1 = 1}$

$x_2 = x_3$

Stavimo npr $x_2 = x_3 = 0$

$I(e,f)$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ je regularna. Za $(f) = \{1+t, t+t^2, t^2\}$ je

$$A(f) = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

(1 bod)

5. Pokažimo prvo da je suma direktna:

(1 bod)

ako je $x \in \mathrm{Im}P \cap \mathrm{Ker}P$, onda je $x = Py$ za neki $y \in V$ i $Px = 0$. Jedno s drugim daje $x = Py = P^2 y = P(Py) = Px = 0$. Time smo pokazali da je suma direktna.

(1 bod)

Dakle, $\mathrm{Im}P \dotplus \mathrm{Ker}P$ je potprostor od V, a zbog direktnosti sume, dimenzija tog potprostora je $r(P) + d(P)$. Iz teorema o rangu i defektu slijedi da je $\dim(\mathrm{Im}P \dotplus \mathrm{Ker}P) = \dim V$. Stoga je $V = \mathrm{Im}P \dotplus \mathrm{Ker}P$.

Kako su $\{a_1, a_2, \ldots, a_r\}$ i $\{b_1, b_2, \ldots, b_d\}$ baze za $\mathrm{Im}P$ i $\mathrm{Ker}P$ i kako je $V = \mathrm{Im}P \dotplus \mathrm{Ker}P$, to je $d = \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_d\}$ zaista baza prostora V

Uočimo sada da je za $i \in \{1, \ldots, r\}$

$a_i = Pc_i$ za neki $c_i \in V$ $\Rightarrow$ $Pa_i = P^2 c_i = Pc_i = a_i$, za $i \in \{1, \ldots, r\}$. Nadalje, $Pb_i = 0$ $\forall$ $i \in \{1, \ldots, d\}$

Stoga je

$$[P]_d^d = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(1 bod)

(1 bod)

Sada je $k_P(\lambda) = \det(P - \lambda I) = (1-\lambda)^r (-\lambda)^d$

$\Rightarrow \sigma(P) = \{0, 1\}$. $a(1) = r = \dim \operatorname{Im} P$

$a(0) = d = \dim \operatorname{Ker} P$

(1 bod)

Očito je $\operatorname{Im} P \subseteq V_P(1) \Rightarrow r \leq g(1) \leq a(1) = r$

$\operatorname{Ker} P \subseteq V_P(0) \Rightarrow d \leq g(0) \leq a(0) = d$

Prema tome, $V_P(1) = \operatorname{Im} P$ i $V_P(0) = \operatorname{Ker} P$.