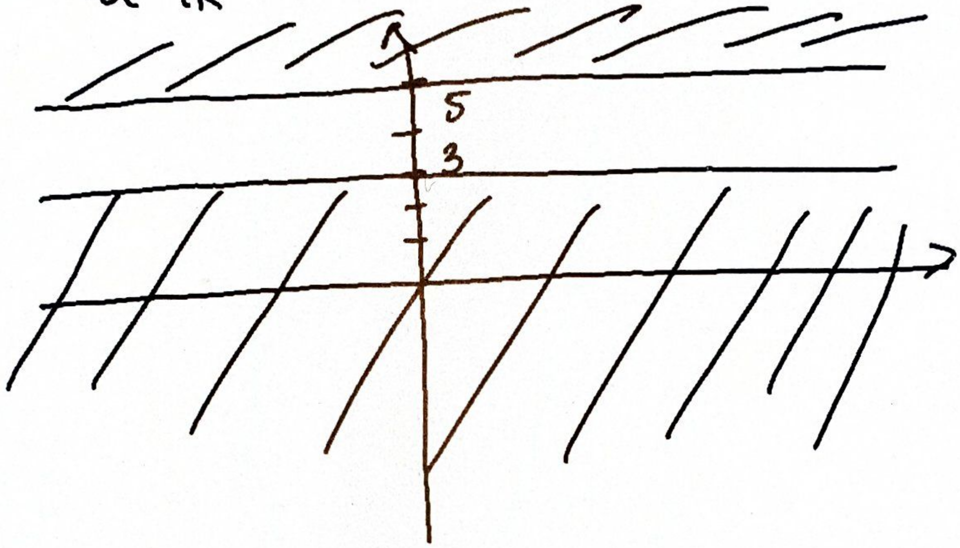
2. a) $A = \{(x,y) \in \mathbb{R}^2 \mid y \in \langle -\infty, 3] \cup [5, +\infty\rangle\}$

u $\mathbb{R}^2$



A je nepovezan. Uzmimo npr.

$U = \mathbb{R} \times \langle -\infty, 4\rangle$, $V = \mathbb{R} \times \langle 4, +\infty\rangle$

U i V su otvoreni, vrijedi

(i) $U \cap A = \mathbb{R} \times \langle -\infty, 3] \neq \emptyset$

$V \cap A = \mathbb{R} \times [5, +\infty\rangle \neq \emptyset$

(ii) $A \subseteq U \cup V$

(iii) $\underbrace{U \cap V}_{\emptyset} \cap A = \emptyset$

$\Rightarrow$ A je nepovezan.

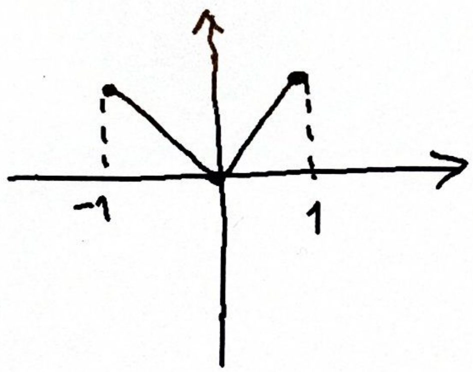
A nije ograničen jer $\forall M>0$

postoji $(x,y) \in A$ t.d. $\sqrt{x^2+y^2} > M$

Uzmimo npr. $(2M,0) \in A$ $\sqrt{4M^2} > M$.

A nije ograničen $\Rightarrow$ A nije kompaktan.

b) $B = \{(x,y) \in \mathbb{R}^2 \mid y = |x|, x \in [-1,1]\}$



B je ograničen. Za $(x,y) \in B$ vrijedi

$\sqrt{x^2+y^2} = \sqrt{x^2+x^2} = 2|x| \leq 2$ pa je

$B \subseteq \overline{K}(0,2) \Rightarrow$ B je ograničen

$f(x,y) = y - |x|$ $\quad$ $P_1(x,y) = x$, $P_2(x,y) = y$

$f = P_2 - |P_1|$ $\quad$ P_1, P_2 neprekidne, $|\cdot|$ neprekidna

$\Rightarrow$ f je neprekidna

$$B = \{ (x,y) \in \mathbb{R}^2 \mid y = |x|,\ x \in [-1,1] \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = 0,\ P_1(x,y) \in [-1,1] \}$$

$$= f^{-1}(\underbrace{\{0\}}_{\text{zatvoren}}) \cap P_1^{-1}(\underbrace{[-1,1]}_{\text{zatvoren}})$$

Kako su f i p neprekidne funkcije, a $\{0\}$ i $[-1,1]$ zatvoreni u $\mathbb{R}$, to su skupovi $f^{-1}(\{0\})$ i $P_1^{-1}([-1,1])$ zatvoreni skupovi pa je i njihov presjek zatvoren skup $\Rightarrow$ B je zatvoren.

Dokažimo da je B putevima povezan, a onda i povezan.

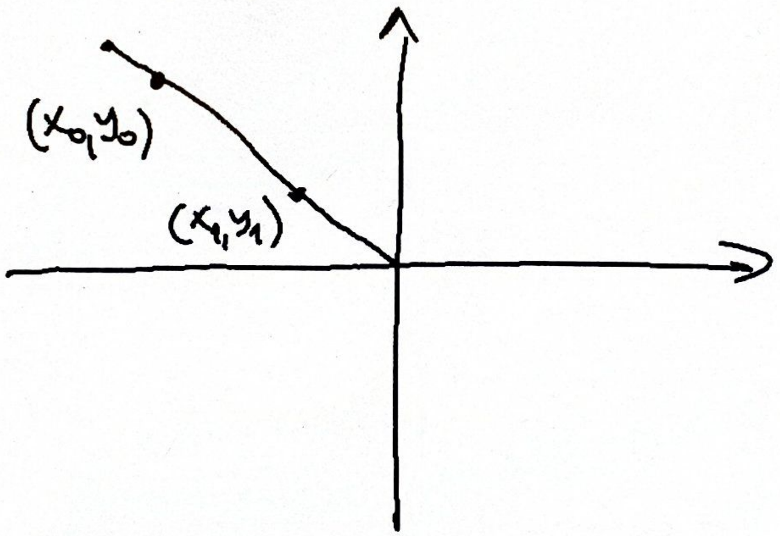
Neka su (x_0, y_0) i (x_1, y_1) bilo koje točke skupa B. ⊗

Ako vrijedi $x_0, x_1 \in [-1, 0]$, $x_0 \leq x_1$, onda

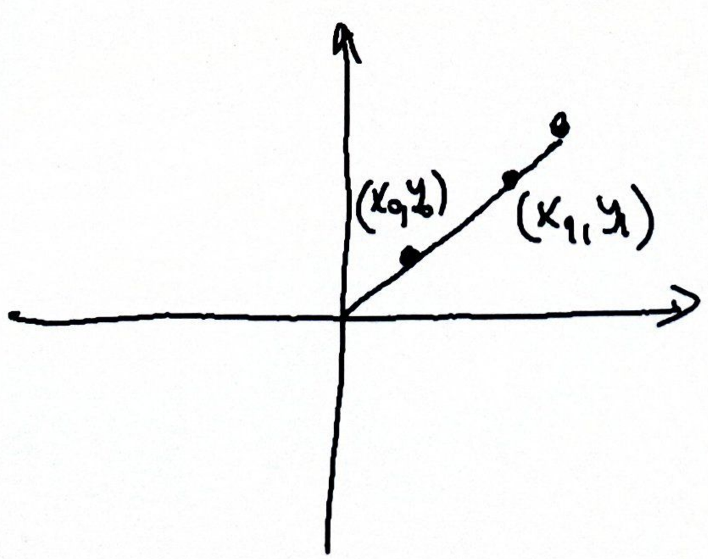
je $\varphi(t) = (1-t)(x_0, y_0) + t(x_1, y_1)$ $t \in [0,1]$

$$= \big((1-t)x_0 + t x_1, (1-t)y_0 + t y_1\big)$$

put u B.



Slično napravimo i ako je $x_0, x_1 \in [0, 1]$



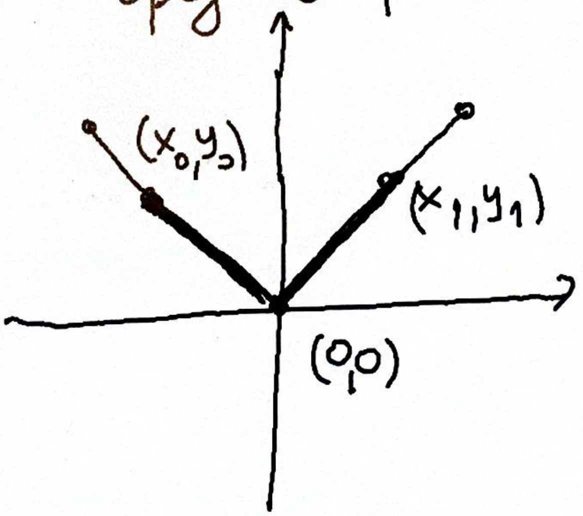
Ako su $(x_0, y_0), (x_1, y_1) \in B$ t.d.

je $x_0 \in [-1, 0]$, $x_1 \in [0, 1]$, onda

prvo spojimo putem točke (x_0, y_0) i $(0,0)$

$$\varphi_1(t) = (1-t)(x_0, y_0) + t(0,0)$$
$$= (1-t)(x_0, y_0),$$

a potom spojimo putem točke $(0,0)$ i (x_1, y_1)



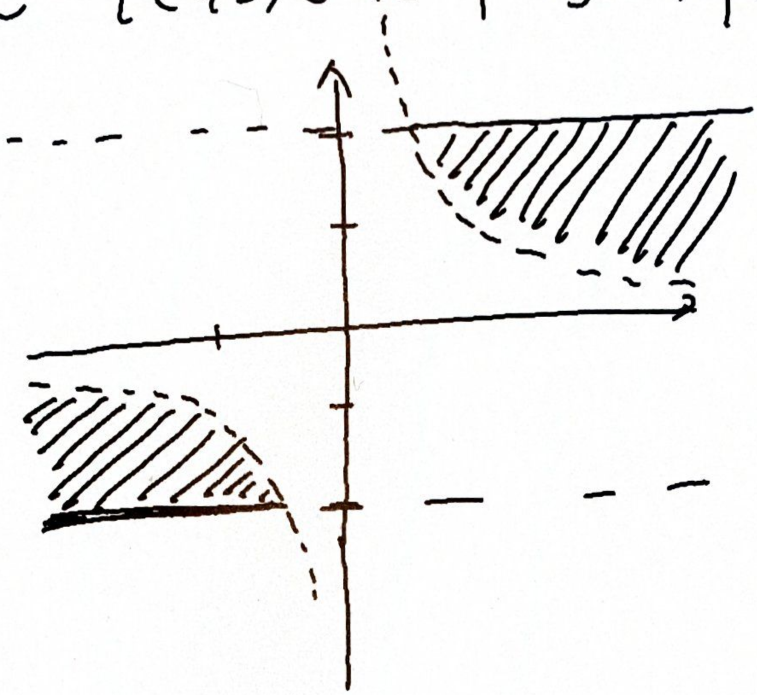
$$\varphi_2(t) = (1-t)(0,0) + t(x_1, y_1)$$
$$= t(x_1, y_1)$$

$[(x_0, y_0), (0,0)] \cup [(0,0), (x_1, y_1)]$ je put

u B od (x_0, y_0) do (x_1, y_1).

B je putevima povezan $\Rightarrow$ B je povezan.

c) $C = \{(x,y) \in \mathbb{R}^2 \mid xy > 1, |y| \leq 2\}$



C je neograničen. Za $M > 1$ je

$\left(M, \frac{2}{M}\right) \in C$ $\left(M \cdot \frac{2}{M} = 2 > 1,\right.$

$\left.\left|\frac{2}{M}\right| < \frac{2}{1} = 2\right)$

$\sqrt{M^2 + \left(\frac{2}{M}\right)^2} > M$

Kako C nije ograničen $\Rightarrow$ C nije kompaktan.

C je nepovezan. Stavimo

$U = \langle 0, +\infty\rangle \times \langle 0, +\infty\rangle$, $V = \langle -\infty, 0\rangle \times \langle -\infty, 0\rangle$

U i V su otvoreni skupovi

⊗

i) $U \cap C = \{(x,y) \in \mathbb{R}^2 : xy > 1,\ |y| \leq 2,$
$x > 0,\ y > 0\}$

$\neq \emptyset$

npr. točka $(2,1) \in U \cap C$

$V \cap C = \{(x,y) \in \mathbb{R}^2 : xy > 1,\ |y| \leq 2,\ x < 0,\ y < 0\}$

$\neq \emptyset$

npr. točka $(-2,-1) \in V \cap C$

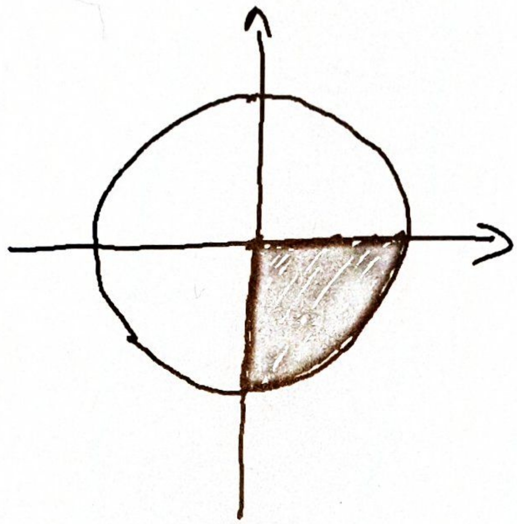
ii) za $(x,y) \in C \Rightarrow xy > 1 > 0$
$\Rightarrow (x,y) \in \langle 0,+\infty\rangle \times \langle 0,+\infty\rangle$
ili
$(x,y) \in \langle -\infty,0\rangle \times \langle -\infty,0\rangle$

$\Rightarrow C \subseteq U \cup V$

iii) $\underbrace{U \cap V}_{\emptyset} \cap C = \emptyset$

Dakle, C je nepovezan.

d) $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1,\ x\geq 0,\ y\le 0\}$



$f(x,y)=x^2+y^2$

$P_1(x,y)=x,\quad P_2(x,y)=y \qquad f=P_1^2+P_2^2$

$$D=\{(x,y)\in\mathbb{R}^2 \mid f(x,y)\in\langle-\infty,1],\ P_1(x,y)\in[0,+\infty\rangle,\ P_2(x,y)\in\langle-\infty,0]\}$$

$$=f^{-1}(\underbrace{\langle-\infty,1]}_{\text{zatvoren}})\cap P_1^{-1}(\underbrace{[0,+\infty\rangle}_{\text{zatvoren}})\cap P_2^{-1}(\underbrace{\langle-\infty,0]}_{\text{zatvoren}})$$

f, P_1, P_2 nepr. $\Rightarrow$ D je zatvoren skup

$D\subseteq \overline{K}(0,1) \Rightarrow D$ je omeđen

Pošto je D omeđen i zatvoren, D je kompaktan

D je putevima povezan.

dokaz: $(x_0, y_0), (x_1, y_1) \in D \implies x_0^2 + y_0^2 \leq 1$

$x_0 \geq 0,\ y_0 \leq 0$

$x_1^2 + y_1^2 \leq 1$

$x_1 \geq 0,\ y_1 \leq 1$

$$\varphi(t) = (1-t)(x_0, y_0) + t(x_1, y_1), \quad t \in [0,1]$$

$$\varphi(0) = (x_0, y_0), \quad \varphi(1) = (x_1, y_1)$$

Dokažimo da je $\varphi(t) \in D$

$$\varphi(t) = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1)$$

$$((1-t)x_0 + tx_1)^2 + ((1-t)y_0 + ty_1)^2$$

$$= (1-t)^2(x_0^2 + y_0^2) + 2(1-t)t(x_0x_1 + y_0y_1)$$

$$+ t^2(x_1^2 + y_1^2)$$

$$\leq (1-t)^2 + 2(1-t)t(x_0x_1 + y_0y_1) + t^2$$

$$= (1-t)^2 + (1-t)t\underbrace{(2x_0x_1+2y_0y_1)}_{\substack{\leq (x_0^2+x_1^2+y_0^2+y_1^2) \\ = (x_0^2+y_0^2)+(x_1^2+y_1^2) \\ \leq 2}} + t^2$$

$$\leq (1-t)^2 + 2(1-t)t + t^2 =$$

$$= ((1-t)+t)^2 = 1$$

Nadalje, kako su $x_0, x_1 \geq 0$, onda je i $(1-t)x_0 + tx_1 \geq 0$

$y_0, y_1 \leq 0 \Rightarrow (1-t)y_0 + ty_1 \leq 0$

$\Rightarrow \varphi(t)$ je put u D

D je putevima povezan $\Rightarrow$ D je povezan.