Zad: U unitarnom prostoru $M_2(\mathbb{R})$ sa skalarnim produktom $\langle X, Y\rangle = tr(Y^t X)$ odredite ortogonalnu projekciju matrice $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ na potprostor Ker tr.

rj: $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, $Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$

$$Y^t X = \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} y_{11}x_{11} + y_{21}x_{21} & y_{11}x_{12} + y_{21}x_{22} \\ y_{12}x_{11} + y_{22}x_{21} & y_{12}x_{12} + y_{22}x_{22} \end{bmatrix}$$

$$\langle X, Y\rangle = tr(Y^t X) = x_{11}y_{11} + x_{12}y_{12} + x_{21}y_{21} + x_{22}y_{22}$$

(Uočite da smo mogli ovo i brže jer je standardni skalarni produkt na $M_2(\mathbb{R})$ dan s $tr(XY^t) = x_{11}y_{11} + x_{12}y_{12} + x_{21}y_{21} + x_{22}y_{22}$, a znamo da je $tr(AB) = tr(BA)$ pa je $tr(Y^t X) = tr(XY^t)$, tako da zapravo imamo standardni skalarni produkt na $M_2(\mathbb{R})$).

$tr: M_2(\mathbb{R}) \to \mathbb{R}$ $\quad \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in Ker(tr) \iff tr \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = 0$

$\iff x_{11}+x_{22}=0 \iff x_{22}=-x_{11} \Rightarrow Ker(tr)=\left\{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}\end{bmatrix};\ x_{11},x_{12},x_{21} \in \mathbb{R}\right\}$

Jedna baza za $Ker(tr)$ je

$$\left\{\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}}_{B_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}}_{B_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}}_{B_3}\right\}$$

Ortonormirat ćemo bazu $\{B_1, B_2, B_3\}$ pa ćemo dobiti bazu $\{e_1, e_2, e_3\}$, a onda ćemo je nadopuniti do ONB za $M_2(\mathbb{R})$ $\{e_1, e_2, e_3, e_4\}$. Tada je (vidjeti vježbe)

$$A = \underbrace{\langle A, e_1\rangle e_1 + \langle A, e_2\rangle e_2 + \langle A, e_3\rangle e_3}_{\in Ker(tr)} + \underbrace{\langle A, e_4\rangle e_4}_{\in Ker(tr)^\perp}$$

GS- postupkom dobijemo da je $e_1 = B_1$, $e_2 = B_2$ (normirani su i okomiti).

Uočite da je $B_3 \perp e_1$ i $B_3 \perp e_2$ pa zapravo samo trebamo normirati $B_3 \Rightarrow e_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$. Stoga je ortogonalna projekcija matrice A na potprostor $Ker(tr)$ dana sa

$$\langle A, e_1\rangle e_1 + \langle A, e_2\rangle e_2 + \langle A, e_3\rangle e_3 = 0\cdot e_1 + 1\cdot e_2 + \frac{1}{\sqrt{2}} e_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2}\end{bmatrix}.$$

Zad: $M \leq \mathbb{R}^4$, $\dim M = 1$, $M = [\{(1,1,1,1)\}]$, P ortogonalni projektor na M. Pronaći ONB za $M^\perp$ i $P(1,0,0,0)$.

Rj: $\mathbb{R}^4 = M \oplus M^\perp$

$$a_1 = (1,1,1,1)$$

$$e_1 = \frac{a_1}{\|a_1\|} = \frac{1}{2}(1,1,1,1)$$

$$x \in M^\perp \iff \langle x, (1,1,1,1)\rangle = 0 \iff x_1 + x_2 + x_3 + x_4 = 0$$
$$\iff x_4 = -x_1 - x_2 - x_3$$

$$M^\perp = \{(x_1, x_2, x_3, -x_1 - x_2 - x_3);\ x_1, x_2, x_3 \in \mathbb{R}\} =$$

$$= [\{\underbrace{(1,0,0,-1)}_{a_2}, \underbrace{(0,1,0,-1)}_{a_3}, \underbrace{(0,0,1,-1)}_{a_4}\}]$$

$$e_2 = \frac{a_2}{\|a_2\|} = \frac{1}{\sqrt{2}}(1,0,0,-1)$$

$$b_3 = a_3 - \langle a_3, e_2\rangle e_2 = (0,1,0,-1) - \frac{1}{2}(1,0,0,-1) =$$
$$= \frac{1}{2}(-1,2,0,-1)$$

$$e_3 = \frac{b_3}{\|b_3\|} = \frac{\frac{1}{2}(-1,2,0,-1)}{\frac{1}{2}\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(-1,2,0,-1)$$

$$b_4 = a_4 - \langle a_4, e_2\rangle e_2 - \langle a_4, e_3\rangle e_3$$

$$= (0,0,1,-1) - \frac{1}{2}(1,0,0,-1) - \frac{1}{6}(-1,2,0,-1)$$

$$= \frac{1}{6}\left((0,0,6,-6) - (3,0,0,-3) + (1,-2,0,1)\right)$$

$$= \frac{1}{6}(-2,-2,6,-2) = \frac{1}{3}(-1,-1,3,-1)$$

$$e_4 = \frac{b_4}{\|b_4\|} = \frac{\frac{1}{3}(-1,-1,3,-1)}{\frac{1}{3}\sqrt{1+1+9+1}} = \frac{1}{2\sqrt{3}}(-1,-1,3,-1)$$

ONB za $M^\perp$ je $\{e_2, e_3, e_4\}$. Stavimo $x = (1,0,0,0)$

$$x = \underbrace{\langle x, e_1\rangle e_1}_{\in M} + \underbrace{\langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3 + \langle x, e_4\rangle e_4}_{\in M^\perp}$$

$$P(x) = \langle x, e_1\rangle e_1$$

$$P(1,0,0,0) = \left\langle (1,0,0,0), \frac{1}{2}(1,1,1,1)\right\rangle \cdot \frac{1}{2}(1,1,1,1) =$$

$$= \frac{1}{4}(1,1,1,1)$$