Zad 2, 2016 a) $f: \langle 0,+\infty\rangle \times \langle 0,+\infty\rangle \to \mathbb{R}^2$ zadana s

$$f(x,y) = \left(\frac{1-\cos(xy)}{x}, \frac{\ln(1+xy)}{y}\right)$$

Može li se f proširiti do neprekidne funkcije na $[0,+\infty\rangle \times [0,+\infty\rangle$? Je li f diferencijabilna u točki $(2,\pi)$? Ako je, odredi joj derivaciju i u toj točki.

Rj:
$$\lim_{\substack{(x,y)\to(x_0,0)\\ x_0\neq 0}} \frac{1-\cos(xy)}{x} = \frac{1-\cos 0}{x_0} = 0 \qquad (1)$$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{x} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{1-\cos(xy)}{xy}}_{\to 0} \cdot \underbrace{y}_{\to 0} = 0 \qquad (2)$$

$$\lim_{t\to 0} \frac{1-\cos t}{t} = \lim_{t\to 0} \frac{(1-\cos t)'}{t'} = \lim_{t\to 0} \frac{\sin t}{1} = 0$$

$$\text{Iz (1) \& (2)} \Rightarrow \lim_{(x,y)\to(x_0,0)} \frac{1-\cos(xy)}{x} = 0 \quad \forall x_0 \in [0,+\infty\rangle$$

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{y} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{\ln(1+xy)}{xy}}_{\to 1} \cdot \underbrace{x}_{\to 0} = 0 \qquad (3)$$

$$\lim_{t\to 0} \frac{\ln(1+t)}{t} = \lim_{t\to 0} \frac{(\ln(1+t))'}{t'} = \lim_{t\to 0} \frac{\frac{1}{1+t}}{1} = 1$$

$$\lim_{(x,y)\to(x_0,0)} \frac{\ln(1+xy)}{y} = \lim_{(x,y)\to(x_0,0)} \underbrace{\frac{\ln(1+xy)}{xy}}_{\to 1} \cdot \underbrace{x}_{\to x_0} = x_0 \qquad (4)$$

$$\Downarrow$$
$$xy \to 0$$

$$(1) \& (4) \Rightarrow \lim_{(x,y)\to(x_0,0)} f(x,y) = (0, x_0) \quad (*)$$
$$(2) \& (3) \nearrow$$

$$\lim_{(x,y)\to(0,y_0)} \frac{1-\cos(xy)}{x} = \lim_{(x,y)\to(0,y_0)} \underbrace{\frac{1-\cos(xy)}{xy}}_{\to 0} \cdot \underbrace{y}_{\to y_0} = 0$$

$$\lim_{(x,y)\to(0,y_0)} \frac{\ln(1+xy)}{y} = \lim_{(x,y)\to(0,y_0)} \underbrace{\frac{\ln(1+xy)}{xy}}_{\to 1} \cdot \underbrace{x}_{\to 0} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,y_0)} f(x,y) = (0,0)$$

$$\tilde{f}(x,y) = \begin{cases} f(x,y), & (x,y) \in \langle 0,+\infty\rangle \times \langle 0,+\infty\rangle \\ (0,x_0), & (x,y) = (x_0,0) \\ (0,0), & (x,y) = (0,y_0) \end{cases}$$

$\tilde{f}$ je neprekidna na $[0,+\infty\rangle \times [0,+\infty\rangle$

U točki $(2,\pi)$:

$$\frac{\partial f_1}{\partial x}(x,y) = \frac{(1-\cos(xy))' \cdot x - (1-\cos(xy))}{x^2} = \frac{(\sin(xy)) \cdot xy - 1 + \cos(xy)}{x^2}$$

$$\frac{\partial f_1}{\partial y}(x,y) = \frac{1}{x}(x \sin(xy)) = \sin(xy)$$

$$\frac{\partial f_2}{\partial x}(x,y) = \frac{1}{\cancel{y}} \frac{1}{1+xy} \cdot \cancel{y} = \frac{1}{1+xy}$$

$$\frac{\partial f_2}{\partial y}(x,y) = \frac{(\ln(1+xy))' \cdot y - \ln(1+xy)}{y^2} = \frac{1}{y} \cdot \frac{1}{1+xy} \cdot x - \frac{\ln(1+xy)}{y^2}$$

Uočimo da f ima parcijalne derivacije i neprekidne su na $\langle 0,+\infty\rangle \times \langle 0,+\infty\rangle$ pa je f derivabilna na $\langle 0,+\infty\rangle \times \langle 0,+\infty\rangle$, a onda i u $\langle 2,\pi\rangle$.

$$f'(2,\pi) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(2,\pi) & \frac{\partial f_1}{\partial y}(2,\pi) \\ \frac{\partial f_2}{\partial x}(2,\pi) & \frac{\partial f_2}{\partial y}(2,\pi) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{1+2\pi} & \frac{2}{\pi}\cdot\frac{1}{1+2\pi} - \frac{\ln(1+2\pi)}{\pi^2} \end{bmatrix}$$