2019.

3.b) Neka je $x \in \mathbb{R}$ i $A \in M_3(\mathbb{R})$ matrica za koju je $tr(A) = det(A) = x$ i x je njena jedina realna svojstvena vrijednost. Koje su joj preostale svojstvene vrijednosti?

Rj:

$\lambda_1 = x \qquad \lambda_2 = a + ib, \ \lambda_3 = a - ib, \ a, b \in \mathbb{R}$

$trA = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow x = x + 2a \Rightarrow \boxed{a=0}$

$detA = \lambda_1 \lambda_2 \lambda_3 = x \cdot (a^2 + b^2) = x \Rightarrow ((a^2+b^2)-1)x = 0$

$\Rightarrow (b-1)(b+1)x = 0$

za $x = 0 \Rightarrow \lambda_2, \lambda_3 = \pm bi, \ b \neq 0$

za $x \neq 0 \Rightarrow b = \pm 1 \Rightarrow \lambda_2, \lambda_3 = \pm i$

2017. (4.) Zadan je operator $A: \mathbb{R}^3 \to \mathbb{R}^3$ formulom

$$A(x_1,x_2,x_3) = (2ax_1,\ ax_1+x_3,\ x_1-2x_2+3x_3)$$

U ovisnosti o parametru $a \in \mathbb{R}$ odredite postoji li baza u kojoj operator A ima dijagonalni matrični prikaz. Kada takav prikaz postoji, odredite dijagonalni prikaz i (neku) bazu u kojoj se postiže.

rj:
$$A(e) = \begin{bmatrix} 2a & 0 & 0 \\ a & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix}$$

$$k_A(\lambda) = \begin{vmatrix} 2a-\lambda & 0 & 0 \\ a & -\lambda & 1 \\ 1 & -2 & 3-\lambda \end{vmatrix} = (2a-\lambda)\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} =$$

$$= (2a-\lambda)\left(-\lambda(3-\lambda)-(-2)\right) = (2a-\lambda)(\lambda^2-3\lambda+2)$$

$$= (2a-\lambda)(\lambda-2)(\lambda-1)$$

(1°) $2a \notin \{1,2\}$, tj. $a \notin \{\frac{1}{2}, 1\}$

Tada je $\sigma(A) = \{1, 2, 2a\}$

$$a(1) = a(2) = a(2a) = 1$$

$$\left.\begin{array}{l} 1 \leq g(1) \leq a(1) = 1 \Rightarrow g(1) = a(1) = 1 \\ 1 \leq g(2) \leq a(2) = 1 \Rightarrow g(2) = a(2) = 1 \\ 1 \leq g(2a) \leq a(2a) = 1 \Rightarrow g(2a) = a(2a) = 1 \end{array}\right\} \Rightarrow A \text{ se može dijagonalizirati}$$

• $V_A(1) = \mathrm{Ker}(A - I)$

$$(A-I)(e) = \begin{bmatrix} 2a-1 & 0 & 0 \\ a & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{array}{l} /:(2a-1) \\ /\cdot(-2) \\ \swarrow \oplus \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 1 \\ 1-2a & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(x_1, x_2, x_3) \in \mathrm{Ker}(A-I) \iff x_1 = 0 \ \& \ -x_2 + x_3 = 0$$

$$\iff (x_1, x_2, x_3) = (0, t, t)$$

$$V_A(1) = [\{(0,1,1)\}]$$

• $V_A(2) = \mathrm{Ker}(A - 2I)$

$$(A-2I)(e) = \begin{bmatrix} 2a-2 & 0 & 0 \\ a & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{array}{l} /:(2a-2) \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ a & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{array}{l} /\cdot(-a) \quad /\cdot(-1) \\ \swarrow \oplus \\ \oplus \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(x_1, x_2, x_3) \in \mathrm{Ker}(A-2I) \iff x_1 = 0 \ \& \ -2x_2 + x_3 = 0$$

$$\iff (x_1, x_2, x_3) = (0, t, 2t)$$

$$V_A(2) = [\{(0,1,2)\}]$$

- $V_A(2a) = \mathrm{Ker}(A - 2aI)$

$$A - 2aI = \begin{bmatrix} 0 & 0 & 0 \\ a & -2a & 1 \\ 1 & -2 & 3-2a \end{bmatrix} /\cdot(-a) \oplus \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2a^2-3a+1 \\ 1 & -2 & 3-2a \end{bmatrix}$$

$$2a^2-3a+1 = 2a^2-2a-a+1$$
$$= 2a(a-1)-(a-1)$$
$$= (2a-1)(a-1)$$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (2a-1)(a-1) \\ 1 & -2 & 3-2a \end{bmatrix} /: (2a-1)(a-1)$$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3-2a \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$(x_1, x_2, x_3) \in \mathrm{Ker}(A-2aI) \iff x_3 = 0 \ \& \ x_1 - 2x_2 = 0$$
$$\iff (x_1, x_2, x_3) = (2t, t, 0)$$

$$V_A(2a) = [\{(2,1,0)\}]$$

$$(f) = \{(0,1,1), (0,1,2), (2,1,0)\}$$

$$A(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2a \end{bmatrix}$$

(2.) $a=\frac{1}{2}$, tj. $2a=1$ $\qquad \sigma(A)=\{1,2\}$

$a(1)=2, \quad a(2)=1$

$1 \leq g(2) \leq a(2)=1 \implies g(2)=a(2)=1$

$$g(1)=d(A-I)=d\left(\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}\right)=3-r\left(\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}\right)=3-1=2$$

$g(1)=a(1)=2$

A se može dijagonalizirati

- $V_A(1)=Ker(A-I) \quad \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$

$(x_1,x_2,x_3) \in Ker(A-I) \iff x_1-2x_2+2x_3=0$

$\iff (x_1,x_2,x_3)=(2s-2t,s,t)$

$V_A(1)=[\{(2,1,0),(-2,0,1)\}]$

- $V_A(2)=Ker(A-2I)$

$$A-2I=\begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{2} & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(x_1,x_2,x_3) \in Ker(A-2I) \iff x_1=0$ & $-2x_2+x_3=0$

$\iff (x_1,x_2,x_3)=(0,t,2t)$

$V_A(2)=[\{(0,1,2)\}]$

$(f) = \{(2,1,0), (-2,0,1), (0,1,2)\}$

$$A(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. $a = 1$, tj. $2a = 2$ $\quad \sigma(A) = \{1, 2\}$

$a(2) = 2, \quad a(1) = 1$

$1 \leq g(1) \leq a(1) = 1 \Rightarrow g(1) = a(1) = 1$

$$g(2) = d(A - 2I) = d \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} = 3 - 1 = 2$$

Može se dijagonalizirati

$$V_A(1) = Ker(A - I) \qquad A - I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_A(1) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0, \ -x_2 + x_3 = 0\}$$
$$= [\{(0,1,1)\}]$$

$$V_A(2) = Ker(A - 2I) \qquad A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_A(2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; \ x_1 - 2x_2 + x_3 = 0\}$$
$$= [\{(2,1,0), (-1,0,1)\}]$$

$$(f) = \{(0,1,1), (2,1,0), (-1,0,1)\} \qquad A(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\boxed{2018}$ 4. Neka je $A \in M_3(\mathbb{R})$ matrica t.d. $\det A = 1$

a) Ako je $\frac{-1+\sqrt{-3}}{2}$ jedan korijen svojstvenog polinoma od A, odredite preostale korijene.

b) Neka je $A^4 = aA^2 + bA + cI$. Odredite a, b, c koristeći Hamilton–Cayleyev teorem.

rj: a) $k_A(\lambda)$ je polinom 3. stupnja s realnim koeficijentima.

$\frac{-1+\sqrt{3}i}{2}$ je nultočka od $k_A(\lambda) \Rightarrow \frac{-1-\sqrt{3}i}{2}$ je nultočka od $k_A(\lambda)$.

$$\boxed{\lambda_1 = \frac{-1+\sqrt{3}i}{2}, \quad \lambda_2 = \frac{-1-\sqrt{3}i}{2}} \qquad \lambda_3 = \text{treća nultočka}$$

$$\det A = \lambda_1 \lambda_2 \lambda_3 \Rightarrow 1 = \frac{1}{4}(1+3)\cdot \lambda_3 \Rightarrow \boxed{\lambda_3 = 1}$$

b) $$k_A(\lambda) = \left(\frac{-1+\sqrt{3}i}{2} - \lambda\right)\left(\frac{-1-\sqrt{3}i}{2} - \lambda\right)(1-\lambda)$$

$$= -\lambda^3 + \left(-1 - \frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right)\lambda + 1$$

$$= -\lambda^3 + 1$$

$$k_A(A) = 0 \Rightarrow -A^3 + I = 0 \Rightarrow \boxed{A^3 = I}$$

$$A^4 = A^3 \cdot A = A \Rightarrow A = aA^2 + bA + cI$$

$$\Rightarrow aA^2 + (b-1)A + cI = 0$$

$\sigma(A) = \left\{ \frac{-1 \pm \sqrt{3}i}{2}, 1 \right\} \Rightarrow A$ se dijagonalizira u nekoj bazi (f).

$$A(f) = \begin{bmatrix} \frac{-1+\sqrt{3}i}{2} & & \\ & \frac{-1-\sqrt{3}i}{2} & \\ & & 1 \end{bmatrix}$$

$$\Rightarrow (aA^2 + (b-1)A + cI)(f) = \begin{bmatrix} a\left(\frac{-1+\sqrt{3}i}{2}\right)^2 + (b-1)\frac{-1+\sqrt{3}i}{2} + c & 0 & 0 \\ 0 & a\left(\frac{-1-\sqrt{3}i}{2}\right)^2 + (b-1)\frac{-1-\sqrt{3}i}{2} + c & 0 \\ 0 & 0 & a+(b-1)+c \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{a}{2}(1+\sqrt{3}i) + \frac{b-1}{2}(-1+\sqrt{3}i) + c & 0 & 0 \\ 0 & -\frac{a}{2}(1-\sqrt{3}i) + (b-1)\frac{-1-\sqrt{3}i}{2} + c & 0 \\ 0 & 0 & a+(b-1)+c \end{bmatrix} = 0$$

$$\Rightarrow \ldots \Rightarrow a = 0, \; b = 1, \; c = 0$$

2018. (2.) Neka je operator $A \in L(\mathbb{R}^3)$ dan matricom

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix}$$

u kanonskoj bazi.

a) Može li se A dijagonalizirati?

b) Odredite matrični prikaz od A^{20} u kanonskoj bazi

Rj: a) $$k_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & 2-\lambda & -3 \\ 3 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 3 & -3 \\ 3 & -2-\lambda \end{vmatrix}$$

$$-1 \cdot \begin{vmatrix} 3 & 2-\lambda \\ 3 & 1 \end{vmatrix} =$$

$$= (2-\lambda)(\lambda^2-4+3) - (-6-3\lambda+9) - (3-6+3\lambda)$$

$$= (2-\lambda)(\lambda^2-1) + 3(\lambda-1) - 3(\lambda-1)$$

$$= (2-\lambda)(\lambda-1)(\lambda+1)$$

$\sigma(A) = \{1, -1, 2\}$

$1 \leq g(1) \leq a(1) = 1 \Rightarrow g(1) = a(1) = 1$

$1 \leq g(-1) \leq a(-1) = 1 \Rightarrow g(-1) = a(-1) = 1$

$1 \leq g(2) \leq a(2) = 1 \Rightarrow g(2) = a(2) = 1$

A se može dijagonalizirati

• $A-I=\begin{bmatrix}1&1&-1\\3&1&-3\\3&1&-3\end{bmatrix}\begin{matrix}\\ /\cdot(-1)\\ \oplus\end{matrix} \sim \begin{bmatrix}1&1&-1\\3&1&-3\\0&0&0\end{bmatrix}\begin{matrix}/\cdot(-3)\\ \oplus\\ \\ \end{matrix} \sim \begin{bmatrix}1&1&-1\\0&-2&0\\0&0&0\end{bmatrix}\begin{matrix}\\ /:(-2)\\ \\ \end{matrix}$

$\sim \begin{bmatrix}1&1&-1\\0&1&0\\0&0&0\end{bmatrix}\begin{matrix}\oplus\\ /\cdot(-1)\\ \\ \end{matrix} \sim \begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}$

$V_A(1) = Ker(A-I) = [\{(1,0,1)\}]$

• $A-2I=\begin{bmatrix}0&1&-1\\3&0&-3\\3&1&-4\end{bmatrix}\begin{matrix}\\ /:3\\ \\ \end{matrix} \sim \begin{bmatrix}0&1&-1\\1&0&-1\\3&1&-4\end{bmatrix}\begin{matrix}/\cdot(-2)\\ /\cdot(-2)\\ \oplus\end{matrix}$

$\sim \begin{bmatrix}0&1&-1\\1&0&-1\\1&-1&0\end{bmatrix}\begin{matrix}/\cdot\\ /\cdot(-1)\\ \oplus\ \oplus\end{matrix} \sim \begin{bmatrix}0&1&-1\\1&0&-1\\0&0&0\end{bmatrix}$

$V_A(2) = Ker(A-2I) = [\{(1,1,1)\}]$

• $A+I=\begin{bmatrix}3&1&-1\\3&3&-3\\3&1&-1\end{bmatrix}\begin{matrix}\\ /:3\\ \\ \end{matrix} \sim \begin{bmatrix}3&1&-1\\1&1&-1\\3&1&-1\end{bmatrix}\begin{matrix}\\ /\cdot(-1)\\ \\ \end{matrix} \sim \begin{bmatrix}2&0&0\\1&1&-1\\2&0&0\end{bmatrix}$

$\sim \begin{bmatrix}1&0&0\\1&1&-1\\0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}$

$V_A(-1) = Ker(A+I) = [\{(0,1,1)\}]$

$$(f) = \{(1,0,1), (1,1,1), (0,1,1)\}$$

$$A(f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A(e) = I(e,f) \cdot A(f) \cdot I(f,e) = I(e,f) \cdot A(f) \cdot I(e,f)^{-1}$$

$$A^{20}(e) = I(e,f) \cdot (A(f))^{20} I(e,f)^{-1} =$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \dots$$

2015. 3. zad. Nađite vrijednosti parametra $\gamma \in \mathbb{R}$ za koje je preslikavanje $L : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ zadano s

$$L(A) = \gamma(2A - A^T - 2(\mathrm{tr} A)I + I) + 2A^T - A - 3I,$$

linearni operator. Za dobivene vrijednosti γ dokažite da je L linearni operator, odredite mu rang i defekt te po jednu bazu za slike i jezgre.

Rj: $L(A+B) = \gamma(2(A+B) - (A+B)^T - 2(\mathrm{tr}(A+B))I + I)$

$$+ 2(A+B)^T - (A+B) - 3I =$$

$$= \gamma(2A - A^T - 2(\mathrm{tr} A)I + I) + 2A^T - A - 3I +$$

$$\gamma(2B - B^T - 2(\mathrm{tr}(B))I) + 2B^T - B$$

$$L(A) + L(B) = \gamma(2A - A^T - 2(\mathrm{tr} A)I + I) + 2A^T - A - 3I$$

$$+ \gamma(2B - B^T - 2(\mathrm{tr} B)I + I) + 2B^T - B - 3I$$

$$L(A+B) = L(A) + L(B) \quad \forall A, B \in M_2(\mathbb{R}) \iff$$

$$\gamma I - 3I = 0 \iff \boxed{\gamma = 3}$$

Dakle, za $\gamma \neq 3$, L nije lin. op. jer nije aditivan.

Za $\gamma = 3$ je L aditivna funkcija. Provjerimo i da je homogena.

Za $\gamma = 3$ $\quad L(A) = 3\left(2A - A^T - 2(\mathrm{tr}A)I + I\right) + 2A^T - A - 3I =$

$$= 5A - A^T - 6(\mathrm{tr}A)I$$

$$L(\alpha A) = 5(\alpha A) - (\alpha A)^T - 6(\mathrm{tr}(\alpha A))I =$$

$$= \alpha \cdot 5A - \alpha \cdot A^T - \alpha \cdot 6(\mathrm{tr}(A))I$$

$$= \alpha\left(5A - A^T - 6(\mathrm{tr}A)I\right) = \alpha \cdot L(A)$$

Dakle, za $\gamma = 3$ je L linearan operator.

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 5a & 5b \\ 5c & 5d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} - 6(a+d)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4a & 5b-c \\ 5c-b & 4d \end{bmatrix} - \begin{bmatrix} 6a+6d & 0 \\ 0 & 6a+6d \end{bmatrix} = \begin{bmatrix} -2a-6d & 5b-c \\ 5c-b & -6a-2d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Ker}L \iff \left.\begin{aligned} -2a-6d &= 0 \\ 5b-c &= 0 \\ 5c-b &= 0 \\ -6a-2d &= 0 \end{aligned}\right\} \iff a=b=c=d=0$$

$$\mathrm{Ker}L = \{0_{M_2(\mathbb{R})}\} \Rightarrow d(L) = 0 \underset{\substack{\text{teorem} \\ \text{o rangu} \\ \text{i defektu}}}{\Rightarrow} r(L) = \dim M_2(\mathbb{R}) - 0 = 4$$

$\Rightarrow \mathrm{Im}L = M_2(\mathbb{R})$ $\quad$ Jedna baza za $\mathrm{Im}A = M_2(\mathbb{R})$ je $\{E_{11}, E_{12}, E_{21}, E_{22}\}$