# Turbulence and Mixing in Holmboe Waves*

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* summary of a paper by the same title submitted to
Journal of Physical Oceanography, November 19th, 2001

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### Abstract

Motivated by the tendency of high Prandtl number fluids to form sharp density interfaces, we investigate the evolution of Holmboe waves in a stratified shear flow via direct numerical simulation. Like their better-known cousins, Kelvin-Helmholtz waves, Holmboe waves lead the flow to a turbulent state in which rapid irreversible mixing takes place. In both cases, significant mixing also takes place prior to the transition to turbulence. Although Holmboe waves grow more slowly than Kelvin-Helmholtz waves, the net amount of mixing that takes place is in fact greater. We conclude that Holmboe instability represents a potentially important source of turbulence in the ocean, and hypothesize that this may be true in general for unstable modes with low growth rates.


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The flow of interest is defined by two homogeneous water masses of different densities separated by a horizontal, sheared interface (as may occur, for example, in an exchange flow or a river outflow). The lower layer is presumed to have the higher density, so that the system is statically stable (figure 1). If the shear across the interface is strong enough to overcome the stabilizing effect of the stratification, instability leads to the development of large-amplitude, wavelike structures that may break and generate turbulence. In the absence of forcing, turbulence eventually decays and the mean flow relaxes to a stable, parallel state. This process results in diapycnal mixing and irreversible potential energy gain.

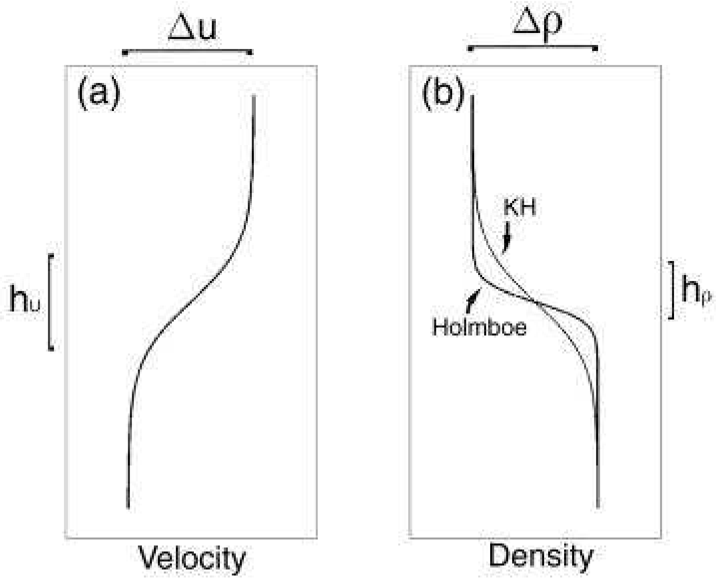
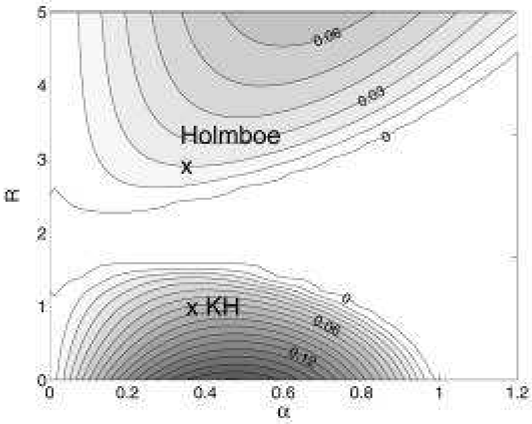



**Figure 1:** Schematic representation of background profiles that exhibit KH and Holmboe instabilities. (a) Background velocity; (b) background density.

We consider a class of unstable initial flows as illustrated in figure 1 such that the velocity difference $\Delta u$, the density difference $\Delta\rho$, and the shear layer thickness $h_U$ are all fixed. Only the stratified layer thickness, $h_\rho$, is allowed to vary.

**Figure 2:** Nondimensional growth rates of unstable modes of the flows shown in figure 1. Growth rates are scaled by $\Delta u / h_U$. The abscissa is the nondimensional wavenumber (scaled by $2 / h_U$ ) and the ordinate is $R \equiv h_U / h_\rho$, the ratio of shear layer depth to stratified layer depth. The "**x**" symbols mark cases to be investigated via DNS.




Adjustment to a stable state may begin though one of two mechanisms, the Kelvin-Helmholtz (KH) instability and the Holmboe instability (Holmboe 1962; Smyth & Winters 2001), depending on the value of $R$ (figure 2). The KH mode is stationary with respect to the center of the shear layer and exhibits relatively large growth rates. The Holmboe mode consists of a pair of disturbances that propagate in opposite directions to form a structure resembling a standing wave. The growth rate of the Holmboe mode is relatively small.

Our objective was to investigate the complete adjustment process for two initial states that represent the KH and Holmboe regimes (crosses on figure 2). To do this, we initialized a direct numerical simulation (DNS) model with the initial profiles shown in figure 1 (plus a weak random noise field to initiate instability). In each case, the simulated flow developed large-amplitude, wavelike structures that became increasingly chaotic and dissipative over time, i.e. it evolved into stratified turbulence (figures 3 and 4). Turbulence then subsided, and the flow relaxed to a stable, parallel state. Details of the KH and Holmboe cases will now be described in turn.

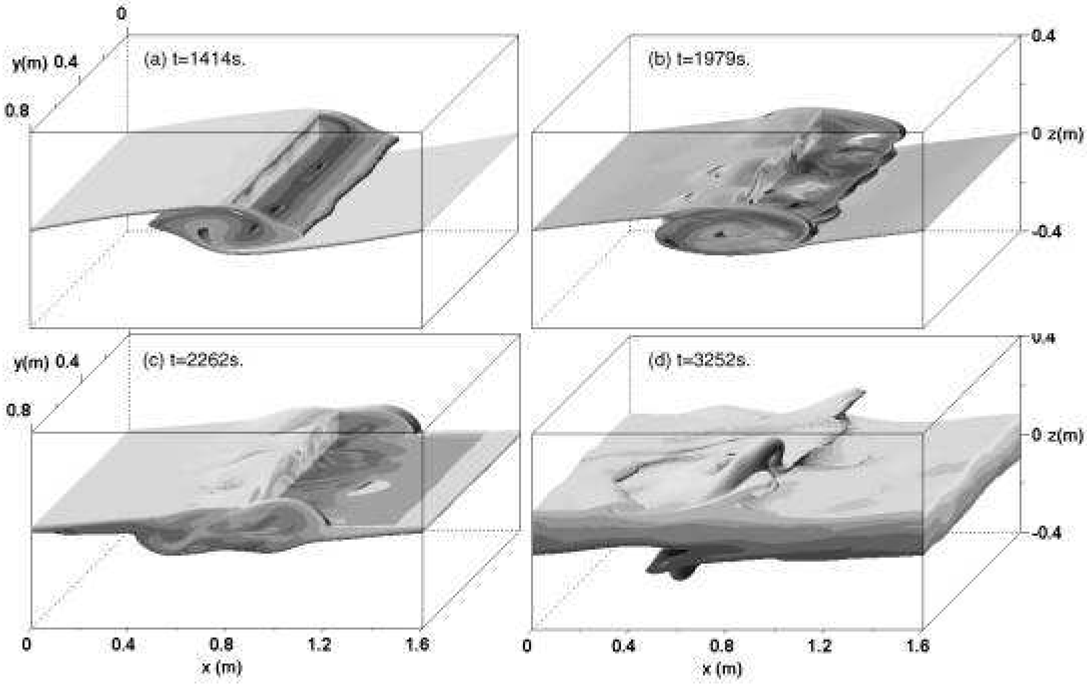



**Figure 3:** Density structure of the KH billow as it approaches maximum amplitude (a,b), as it evolves into a Holmboe wave (c), and after one oscillation period of the Holmboe wave (d). The range of shading from dark to light corresponds to the range of density from high to low, but the correspondence is not exact as additional lighting effects are used to highlight three-dimensionality. The highest and lowest densities are rendered transparent, as is all fluid in the upper right-hand quarter of the billow in (a), (b) and (c).

The initial growth of the KH billow resembled that seen in previous KH experiments (e.g. Smyth &

Moum 2000). Later in the evolution, however, the large-scale structure changed radically. The KH billow (figures 3a, 3b) separated into a pair of oppositely propagating disturbances (figure 3c), which then exhibited the loop-like structures (figure 3d) characteristic of turbulent Holmboe waves (as discussed below). This behavior has been seen in laboratory experiments on KH waves in salt-stratified water (Browand et al. 1973, Hogg & Ivey 2001). This result emphasizes the potential importance of Holmboe waves in the ocean: they tend to occur even when the initial conditions favor KH waves.

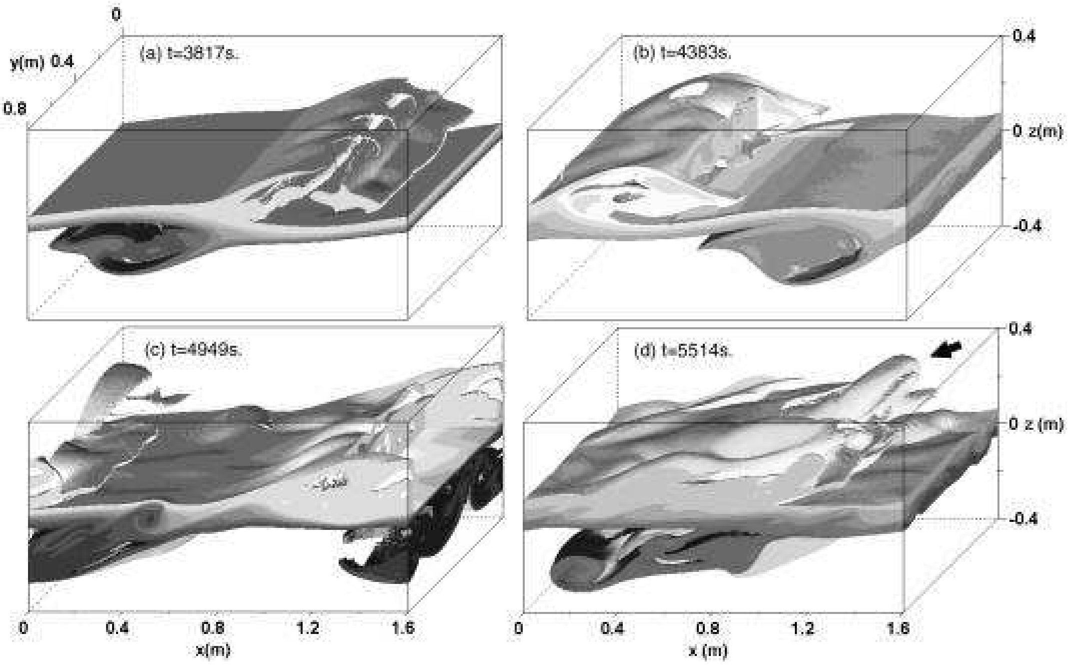
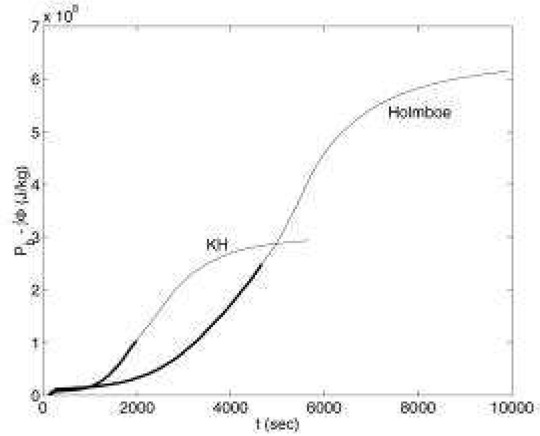



**Figure 4:** Density structure of the Holmboe wave through approximately two oscillation cycles. The range of shading from dark to light corresponds to the range of density from high to low, but the correspondence is not exact as additional lighting effects are used to highlight three-dimensionality. The highest and lowest densities are rendered transparent, as are sections of the upper billow in (b) and (c). The arrow on (d) indicates the loop structure characteristic of the turbulent Holmboe wave.

This exploratory project was designed mainly to answer a single question: “Do Holmboe waves develop significant turbulence?” The low growth rates predicted by linear theory suggest that the answer is “no”, but this did not turn out to be the case. The Holmboe flow developed large-amplitude, oppositely propagating billows, which became distinctly turbulent before they decayed (figure 4). A conspicuous feature of the turbulence arising from Holmboe waves was the appearance of “loops” that resemble hairpin vortices (figure 4d). Detailed analyses have shown that these loops are maintained by a balance between vortex stretching and viscous diffusion, like the Burgers vortices that arise in homogeneous turbulence (e.g. Andreotti 1997). Similar structures develop in KH waves, but because the KH waves are stationary, the loops become entrained into the billows. In this case, the loops are ejected far above (or below) the shear layers, carrying fluid that is then partially mixed into the surrounding fluid, irreversibly raising the potential energy of the flow.

**Figure 5:** Potential energy gain due to fluid motion in the KH and Holmboe cases. Given time, the Holmboe wave accomplishes more work against gravity than does the KH wave. Thick sections of curves represent the preturbulent phase, which contributes significantly to mixing in both cases.



Irreversible potential energy gain is the most useful measure of diapycnal mixing in stratified flow. Figure 5 shows the irreversible potential energy gain as a function of time for the KH and Holmboe cases. The Holmboe case accomplished twice as much mixing as the KH case! To elaborate on this unexpected result, we examine $\Gamma$, the ratio of potential energy gain to viscous dissipation, for the two cases. In previous research, Smyth et al. (2001) showed that this ratio greatly exceeds the accepted value 0.2 for preturbulent KH billows, i.e. that the preturbulent billows convert the kinetic energy of the unstable mean flow to potential energy with very little loss to viscosity. Despite their very different structure, preturbulent Holmboe waves also exhibit large $\Gamma$ in the preturbulent phase. This phase also lasts longer in the Holmboe case than in the KH case.

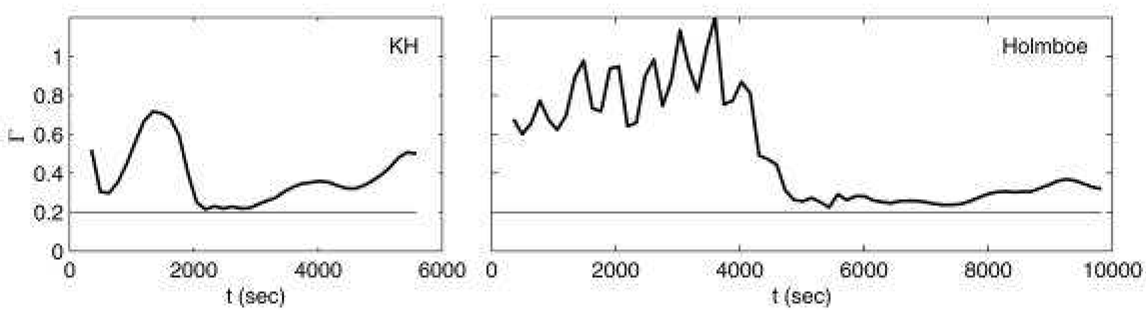



**Figure 6:** Evolution of the flux coefficient for KH and Holmboe cases.

Consideration of these surprising results suggests a new way of looking at mixing in dynamically unstable shear flows. Instability is the mechanism by which an unstable mean flow adjusts to a stable state. This adjustment process invariably involves some degree of mixing, as a part of the kinetic energy shed by the mean flow is converted irreversibly to potential energy. The remaining kinetic energy is lost to viscosity (or, in other situations, radiated as waves). The net amount of mixing depends not on the vigor of the linear instability, but rather on the fraction of mean flow kinetic energy that winds up as potential energy. We now know that preturbulent, wavelike structures may mix with high efficiency, i.e. with minimal loss to viscosity (Smyth et al. 2001). A slowly growing instability may spend more time in this highly efficient, preturbulent state and thus, given sufficient time, mix the fluid more completely than does an instability that extracts mean kinetic energy rapidly but wastes most of it. This distinction is illustrated by the thick segments of the curves in figure 5, which show that a significant fraction of mixing takes place in the preturbulent phase of both the KH and Holmboe events, but that the preturbulent phase lasts longer in the Holmboe case. Holmboe instability mixes effectively, not despite its low growth rate, but because of it.

In summary, ***while positive growth rate is necessary for mixing, a larger growth rate does not indicate more mixing, but rather the opposite***. If this hypothesis is generally valid, then a tremendous amount may be learned about ocean mixing by considering both Holmboe instability and more general classes of shear instability that have been neglected in favor of the rapidly-growing KH wave.

**Acknowledgement**: This work was funded by the National Science Foundation.

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