14 The Geostrophic Momentum Approximation and the Semi-Geostrophic Equations in Pseudo-Height Coordinates

14.1 Geostrophic momentum approximation on an f-plane

In midlatitude cyclones the relative vorticity is often comparable with the Coriolis parameter, particularly in frontal regions. This raises the question of the validity of quasi-geostrophic theory in simulating the life cycle of midlatitude cyclones. We now proceed to develop semi-geostrophic theory. Like quasi-geostrophic theory, semi-geostrophic theory is a filtered theory in that it does not possess solutions corresponding to freely propagating gravity waves. However, semi-geostrophic theory contains less approximations than quasi-geostrophic theory. The additional physics in semi-geostrophic theory is crucial for the simulation of certain nonlinear aspects of cyclones, especially fronts.

Semi-geostrophic theory has two parts—the geostrophic momentum approximation and the transformation to geostrophic coordinates. The geostrophic momentum approximation to the primitive equations (13.13)–(13.17) is

$$\frac{\partial u_g}{\partial t} + u\frac{\partial u_g}{\partial x} + v\frac{\partial u_g}{\partial y} + w\frac{\partial u_g}{\partial z} - fv + \frac{\partial \phi}{\partial x} = 0, \tag{14.1}$$

$$\frac{\partial v_g}{\partial t} + u\frac{\partial v_g}{\partial x} + v\frac{\partial v_g}{\partial y} + w\frac{\partial v_g}{\partial z} + fu + \frac{\partial \phi}{\partial y} = 0, \tag{14.2}$$

$$\frac{g\theta}{\theta_0} = \frac{\partial \phi}{\partial z}, \tag{14.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial (\rho w)}{\rho \partial z} = 0, \tag{14.4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0, \tag{14.5}$$

where

$$fu_g = -\frac{\partial \phi}{\partial y}, \qquad fv_g = \frac{\partial \phi}{\partial x}.$$

These equations form a balanced system in that they cannot describe gravity wave propagation. In comparing the semi-geostropic equations (14.1)–(14.5) with the quasi-geostrophic equations (13.19)–(13.23) we note that (14.1) and (14.2) include momentum advection by the horizontal ageostrophic motion and by the vertical motion. In addition, the thermodynamic equation (14.5) is exact as opposed to the quasi-geostrophic version (13.23).

The semi-geostrophic equations (14.1)–(14.5) possess a very reasonable vector vorticity equation and potential vorticity equation. These are not simple to derive, and the details are given in Appendix F. The three-dimensional vorticity equation is

$$\rho\frac{D}{Dt}\left(\frac{\boldsymbol{\zeta}_g}{\rho}\right) = \left(\boldsymbol{\zeta}_g \cdot \nabla\right)\mathbf{u} - \frac{g}{\theta_0}\mathbf{k} \times \nabla\theta, \tag{14.6}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

and

$$\boldsymbol{\zeta}_g = \left(-\frac{\partial v_g}{\partial z}, \frac{\partial u_g}{\partial z}, f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \left(\frac{1}{f}\frac{\partial\left(u_g, v_g\right)}{\partial\left(y, z\right)}, \frac{1}{f}\frac{\partial (u_g, v_g)}{\partial (z, x)}, \frac{1}{f}\frac{\partial (u_g, v_g)}{\partial (x, y)}\right).$$

The potential vorticity conservation relation is

$$\frac{DP}{Dt} = 0, \tag{14.7a}$$

where

$$P = \frac{1}{\rho}\boldsymbol{\zeta}_g \cdot \nabla\theta. \tag{14.7b}$$

Note that the semi-geostrophic potential vorticity principle (14.7) is identical to the primitive equation potential vorticity principle except that the vector vorticity $\boldsymbol{\zeta}$ is approximated by $\boldsymbol{\zeta}_g$. The approximation could also, therefore, be referred to as the geostrophic potential vorticity approximation. The extra term in the definition of vorticity is necessary for mathematical consistency. However, in most flows of interest it is only a small correction.

Now consider the energy principle associated with the semi-geostrophic equations (14.1)–(14.5). To obtain the kinetic energy principle we multiply (14.1) by u_g, (14.2) by v_g, (14.3) by w, and then sum the results, using the geostrophic relations, to obtain

$$\frac{\partial K_g}{\partial t} + u\frac{\partial K_g}{\partial x} + v\frac{\partial K_g}{\partial y} + w\frac{\partial K_g}{\partial z} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} = \frac{g}{\theta_0}w\theta, \tag{14.8}$$

where $K_g = \frac{1}{2}(u_g^2 + v_g^2)$ is the geostrophic kinetic energy per unit mass. Using the continuity equation (14.4), we can put (14.8) in the flux form

$$\frac{\partial(\rho K_g)}{\partial t} + \frac{\partial[\rho u(K_g + \phi)]}{\partial x} + \frac{\partial[\rho v(K_g + \phi)]}{\partial y} + \frac{\partial[\rho w(K_g + \phi)]}{\partial z} = \frac{g}{\theta_0}w\theta\rho. \tag{14.9}$$

Integrating (14.9) over the entire volume, assuming there are no net fluxes of K_g or ϕ across the boundaries of the domain, we obtain the kinetic energy principle

$$\frac{d}{dt}\iiint \tfrac{1}{2}\left(u_g^2 + v_g^2\right)\rho dxdydz = \frac{g}{\theta_0}\iiint w\theta\rho dxdydz. \tag{14.10}$$

Note that the semi-geostrophic kinetic energy principle (14.9) is different than the quasi-geostrophic kinetic energy principle (13.32), but that the integrated semi-geostrophic form (14.10) is identical to the integrated quasi-geostrophic form (13.33).

To obtain the potential energy principle, we multiply the thermodynamic equation (14.5) by $-(g/\theta_0)z$, which yields

$$\rho\frac{D}{Dt}\left(-\frac{g}{\theta_0}z\theta\right) = -\frac{g}{\theta_0}w\theta\rho. \tag{14.11}$$

In flux form, (14.11) can be written

$$\frac{\partial}{\partial t}\left[\rho\left(-\frac{g}{\theta_0}z\theta\right)\right] + \frac{\partial}{\partial x}\left[\rho u\left(-\frac{g}{\theta_0}z\theta\right)\right] + \frac{\partial}{\partial y}\left[\rho v\left(-\frac{g}{\theta_0}z\theta\right)\right] + \frac{\partial}{\partial z}\left[\rho w\left(-\frac{g}{\theta_0}z\theta\right)\right] = -\frac{g}{\theta_0}w\theta\rho. \tag{14.12}$$

Integrating (14.12) over the entire domain, assuming there are no net fluxes of $z\theta$ across the boundaries of the domain, the potential energy equation becomes

$$\frac{d}{dt}\iiint\left(-\frac{g}{\theta_0}z\theta\right)\rho dxdydz = -\frac{g}{\theta_0}\iiint w\theta\rho dxdydz. \tag{14.13}$$

Note that the energy conversion term on the right hand side of (14.13) is identical, except for sign, to the conversion term on the right hand side of (14.10). Adding the kinetic energy equation (14.10) and the potential energy equation (14.13), we obtain the total energy equation

$$\frac{d}{dt}\iiint\left\{\tfrac{1}{2}\left(u_g^2 + v_g^2\right) - \frac{g}{\theta_0}z\theta\right\}\rho dxdydz = 0. \tag{14.14}$$

According to (14.14), the sum of the mass integrated geostrophic kinetic energy and potential energy is conserved in semi-geostrophic theory.

For practical numerical prediction, the form (14.1)–(14.5) is inconvenient. The fields u_g, v_g, θ cannot be independently predicted by (14.1), (14.2), and (14.5), since u_g, v_g, θ are all related to ϕ through the geostrophic and hydrostatic relations. In fact, only one dependent variable should be predicted, and all others should be diagnosed.

14.2 Geostrophic coordinates

So far we have discussed the primitive equations, the quasi-geostrophic equations and the geostrophic momentum approximation to the primitive equations. These three systems of equations are summarized in Table 14.1. As we noted in the previous section the equations with the geostrophic momentum approximation look very much like the primitive equations. The quasi-geostrophic equations look more approximate in that vertical advection of momentum is neglected, horizontal advection of momentum and potential temperature is done geostrophically, and vertical motion occurs against a standard atmosphere static stability $N^2(z)$. What we will now show is really quite remarkable. *If the horizontal coordinates are transformed to geostrophic coordinates and transformed ageostrophic components are defined, then the geostrophic momentum equations almost become formally identical to the quasi-geostrophic equations.*

We begin by introducing the new independent variables (X, Y, Z, T) defined by

$$(X, Y, Z, T) = \left(x + \frac{v_g}{f}, y - \frac{u_g}{f}, z, t \right). \tag{14.15}$$

The reason for introducing Z and T is that $\partial/\partial Z \neq \partial/\partial z$ and $\partial/\partial T \neq \partial/\partial t$. The independent variables X and Y are called geostrophic coordinates since, using (14.1) and (14.2), we can write

$$\frac{DX}{Dt} = \frac{Dx}{Dt} + \frac{1}{f}\frac{Dv_g}{Dt} = u - u_{ag} = u_g, \tag{14.16}$$

$$\frac{DY}{Dt} = \frac{Dy}{Dt} - \frac{1}{f}\frac{Du_g}{Dt} = v - v_{ag} = v_g. \tag{14.17}$$

Because of (14.16) and (14.17) we can interpret (X, Y) as the position a particle would have if it moved with the geostrophic velocity at every instant. Let us now relate (x, y, z, t) derivatives to (X, Y, Z, T) derivatives:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} \tag{14.18}$$

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} \tag{14.19}$$

$$\frac{\partial}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} \tag{14.20}$$

$$\frac{\partial}{\partial z} = \frac{\partial X}{\partial z}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial z}\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}. \tag{14.21}$$

Equations (14.18)–(14.21) can be combined to obtain

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial T} + \frac{DX}{Dt}\frac{\partial}{\partial X} + \frac{DY}{Dt}\frac{\partial}{\partial Y} + w\frac{\partial}{\partial Z},$$

which, with the help of (14.16) and (14.17), can be written

$$\frac{D}{Dt} = \frac{\partial}{\partial T} + u_g\frac{\partial}{\partial X} + v_g\frac{\partial}{\partial Y} + w\frac{\partial}{\partial Z}. \tag{14.22}$$

Thus, in the new coordinates the horizontal advecting velocity has become geostrophic. Where did the horizontal ageostrophic advection go? Apparently it has become implicit in the coordinate transformation, i.e., the difference in plotting a solution in (X, Y, Z) and (x, y, z) is essentially due to horizontal ageostrophic advection.

Defining

$$a = \frac{1}{f}\frac{\partial v_g}{\partial x},\ b = -\frac{1}{f}\frac{\partial u_g}{\partial x} = \frac{1}{f}\frac{\partial v_g}{\partial y},\ c = -\frac{1}{f}\frac{\partial u_g}{\partial y},\ \alpha = \frac{1}{f}\frac{\partial v_g}{\partial z},\ \beta = -\frac{1}{f}\frac{\partial u_g}{\partial z}\ , \tag{14.23a}$$

or, in terms of ϕ,

$$a = \frac{1}{f^2}\phi_{xx},\ b = \frac{1}{f^2}\phi_{xy},\ c = \frac{1}{f^2}\phi_{yy},\ \alpha = \frac{1}{f^2}\phi_{xz},\ \beta = \frac{1}{f^2}\phi_{yz}\ , \tag{14.23b}$$

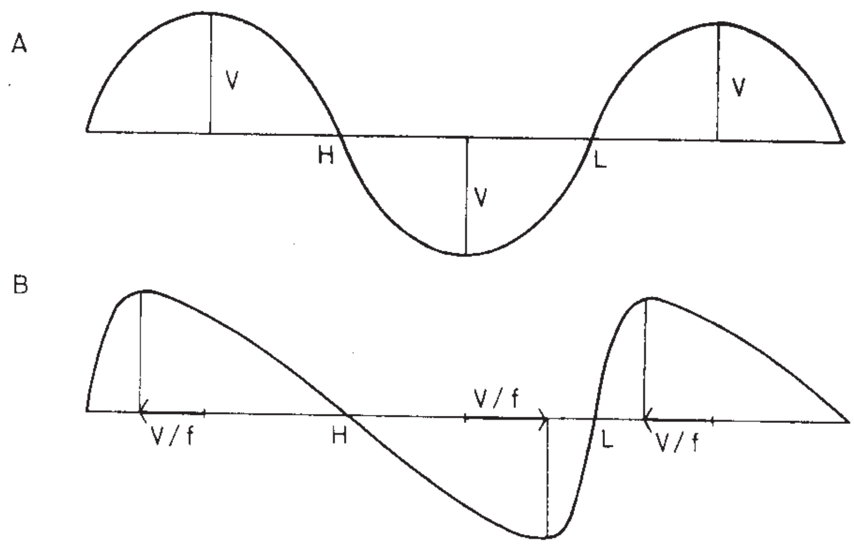

Figure 14.1: Panel A shows a meridional geostrophic wind pattern that is sinusoidal in the geostrophic coordinate X (with the meridional geostrophic wind denoted by V). Panel B shows the transformation to the physical coordinate x, where $x = X - V/f$. Where $V > 0$, there is a shift to the left, and where $V < 0$, there is a shift to the right. As a result, the anticyclonic center H is broadened and weakened, while the cyclonic center L is tightened and strengthened. From Hoskins (1975).

we can write (14.19)–(14.21) as

$$\frac{\partial}{\partial x} = (1+a)\frac{\partial}{\partial X} + b\frac{\partial}{\partial Y} \tag{14.24}$$

$$\frac{\partial}{\partial y} = b\frac{\partial}{\partial X} + (1+c)\frac{\partial}{\partial Y} \tag{14.25}$$

$$\frac{\partial}{\partial z} = \alpha\frac{\partial}{\partial X} + \beta\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z} \; . \tag{14.26}$$

Taking certain combinations of (14.24)–(14.26) we can show that the inverse transformation is

$$J\frac{\partial}{\partial X} = (1+c)\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} \tag{14.27}$$

$$J\frac{\partial}{\partial Y} = -b\frac{\partial}{\partial x} + (1+a)\frac{\partial}{\partial y} \tag{14.28}$$

$$J\frac{\partial}{\partial Z} = -\left[\alpha(1+c) - \beta b\right]\frac{\partial}{\partial x} - \left[\beta(1+a) - \alpha b\right]\frac{\partial}{\partial y} + J\frac{\partial}{\partial z} \; , \tag{14.29}$$

where J, the Jacobian of the transformation, is the nondimensional vertical component of absolute vorticity, i.e.

$$J = (1+a)(1+c) - b^2 = \frac{\partial(X,Y)}{\partial(x,y)} = \frac{1}{f}\mathbf{k}\cdot\boldsymbol{\zeta}_g = \frac{\zeta}{f}. \tag{14.30}$$

Using (14.23a) we can write

$$-\left[\alpha\,(1+c) - \beta b\right] = -\frac{1}{f}\frac{\partial v_g}{\partial z} + \frac{1}{f^2}\frac{\partial\,(u_g, v_g)}{\partial\,(y,z)} = \frac{\partial(X,Y)}{\partial(y,z)} = \frac{\xi}{f}, \tag{14.31}$$

$$-\left[\beta\,(1+a) - \alpha b\right] = \frac{1}{f}\frac{\partial u_g}{\partial z} + \frac{1}{f^2}\frac{\partial\,(u_g, v_g)}{\partial(z,x)} = \frac{\partial(X,Y)}{\partial(z,x)} = \frac{\eta}{f} \; . \tag{14.32}$$

Equations (14.29)–(14.32) imply that

$$fJ\frac{\partial}{\partial Z} = \zeta\frac{\partial}{\partial Z} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z} = \boldsymbol{\zeta}_g \cdot \nabla. \tag{14.33}$$

In other words the geostrophic space operator $fJ(\partial/\partial Z)$ corresponds in physical space to a derivative along the absolute vorticity vector.

Defining

$$\Phi = \phi + \tfrac{1}{2}\left(u_g^2 + v_g^2\right) \tag{14.34}$$

and using (14.27) and (14.23a) we can write

$$\begin{aligned} J\frac{\partial \Phi}{\partial X} &= \left[(1+c)\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right]\left[\phi + \tfrac{1}{2}\left(u_g^2 + v_g^2\right)\right] \\ &= (1+c)\left(\frac{\partial \phi}{\partial x} - u_g f b + v_g f a\right) - b\left(-f u_g - u_g f c + v_g f b\right) \\ &= \left[(1+a)(1+c) - b^2\right]\frac{\partial \phi}{\partial x}\,, \end{aligned}$$

or, using (14.30),

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \phi}{\partial x}. \tag{14.35}$$

Similarly, we can show that

$$\frac{\partial \Phi}{\partial Y} = \frac{\partial \phi}{\partial y}. \tag{14.36}$$

Using (14.26), (14.23a), and (14.34)–(14.36) we can write

$$\frac{\partial \Phi}{\partial Z} + \alpha\frac{\partial \Phi}{\partial X} + \beta\frac{\partial \Phi}{\partial Y} = \frac{\partial \phi}{\partial z} + u_g\frac{\partial u_g}{\partial z} + v_g\frac{\partial v_g}{\partial z},$$

or

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial \phi}{\partial z}. \tag{14.37}$$

We can summarize (14.35)–(14.37) by writing

$$\left(f v_g, -f u_g, \frac{g}{\theta_0}\theta\right) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \left(\frac{\partial \Phi}{\partial X}, \frac{\partial \Phi}{\partial Y}, \frac{\partial \Phi}{\partial Z}\right). \tag{14.38}$$

In other words the geostrophic and hydrostatic relations take the same form in geostrophic space (X, Y, Z) as in physical space (x, y, z).

Now suppose we define the new ageostrophic components

$$\left(u_{ag}^*,\, v_{ag}^*,\, w^*\right) = \left(u_{ag} + \frac{w}{f^2}\Phi_{XZ},\, v_{ag} + \frac{w}{f^2}\Phi_{YZ},\, \frac{w}{J}\right). \tag{14.39}$$

Then, using (14.22), we can write (14.1) and (14.2) as

$$\mathcal{D}_g u_g - f v_{ag}^* = 0, \tag{14.40}$$

$$\mathcal{D}_g v_g + f u_{ag}^* = 0, \tag{14.41}$$

where

$$\mathcal{D}_g = \frac{\partial}{\partial T} + u_g\frac{\partial}{\partial X} + v_g\frac{\partial}{\partial Y}. \tag{14.42}$$

Using (14.22) in the thermodynamic equation (14.5) we obtain

$$\mathcal{D}_g \theta + w\frac{\partial \theta}{\partial Z} = 0.$$

The units on the potential vorticity (14.7b) are PVU. In the geostrophic coordinates the potential vorticity will play the role of a static stability. It is thus useful to define $q_g = [g/(\rho f \theta_0)]\boldsymbol{\zeta}_g \cdot \nabla\theta = [g/(f\theta_0)]P$ and using (14.33) to write $q_g = [g/(\rho\theta_0)]J(\partial\theta/\partial Z)$, we can then write the thermodynamic equation as

$$\mathcal{D}_g\theta + \frac{\theta_0}{g}q_g\rho w^* = 0. \tag{14.43}$$

The form of (14.43) is the same as in quasi-geostrophic theory except ρq_g has replaced N^2.

The last equation we want to transform to geostrophic space is the continuity equation. To do this we start from the vertical component of the vorticity equation. Dotting $\mathbf{k}$ into (14.6) and using (14.33) we obtain

$$\frac{D}{Dt}\left(\mathbf{k}\cdot\boldsymbol{\zeta}_g\right) = fJ\frac{\partial(\rho w)}{\rho\partial Z},$$

or, because of (14.30)

$$\begin{aligned}
\frac{DJ}{Dt} &= J\frac{\partial(\rho w)}{\rho\partial Z}, \\
\mathcal{D}_gJ + w\frac{\partial J}{\partial Z} &= J\frac{\partial(\rho w)}{\rho\partial Z}, \\
\frac{\mathcal{D}_gJ}{J^2} &= \frac{\partial}{\rho\partial Z}\left(\frac{\rho w}{J}\right), \\
\mathcal{D}_gJ^{-1} &= -\frac{\partial(\rho w^*)}{\rho\partial Z}. \qquad (14.44)
\end{aligned}$$

From (14.11) and (14.29) we have

$$\mathbf{k}\cdot\boldsymbol{\zeta}_g = fJ = f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\left(\frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial y} - \frac{\partial v_g}{\partial x}\frac{\partial u_g}{\partial y}\right).$$

As discussed at the end of section 14.1 the last term is a small correction which can be neglected at the level of the geostrophic momentum approximation so that

$$fJ = f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}. \tag{14.45}$$

Now application of (14.27) and (14.30) yields

$$\begin{aligned}
\frac{J}{f}\frac{\partial v_g}{\partial X} &= (1+c)\,\frac{1}{f}\frac{\partial v_g}{\partial x} - b\frac{1}{f}\frac{\partial v_g}{\partial y} = (1+c)\,a - b^2 \\
J - \frac{J}{f}\frac{\partial v_g}{\partial X} &= (1+a)\,(1+c) - b^2 - a\,(1+c) + b^2 = 1 + c,
\end{aligned}$$

or

$$J\left(1 - \frac{1}{f}\frac{\partial v_g}{\partial X}\right) = 1 + c. \tag{14.46}$$

Similarly,

$$J\left(1 + \frac{1}{f}\frac{\partial u_g}{\partial Y}\right) = 1 + a, \tag{14.47}$$

and

$$\frac{J}{f}\frac{\partial v_g}{\partial Y} = -\frac{J}{f}\frac{\partial u_g}{\partial X} = b. \tag{14.48}$$

Using (14.30) we can write (14.45) as

$$fJ = f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = f\,(1 + a + c)\ .$$

Using (14.46) and (14.47) this becomes

$$J = J\left(1 + \frac{1}{f}\frac{\partial u_g}{\partial Y}\right) + J\left(1 - \frac{1}{f}\frac{\partial v_g}{\partial X}\right) - 1\ ,$$

or, collecting the terms with J

$$J = \frac{f}{f - \left(\frac{\partial v_g}{\partial X} - \frac{\partial u_g}{\partial Y}\right)} = \frac{f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}}{f}\ . \tag{14.49}$$

Since J is the dimensionless vertical component of absolute vorticity, as $\partial v_g/\partial X - \partial u_g/\partial Y$ approaches f the vertical component of absolute vorticity approaches infinity. Returning now to the derivation of the transformed continuity equation, we substitute (14.49) into (14.44) to obtain

$$\frac{1}{f}\mathcal{D}_g\left(-\frac{\partial v_g}{\partial X} + \frac{\partial u_g}{\partial Y}\right) + \frac{\partial(\rho w^*)}{\rho \partial Z} = 0,$$

or

$$\frac{1}{f}\left(-\frac{\partial}{\partial X}\mathcal{D}_g v_g + \frac{\partial}{\partial Y}\mathcal{D}_g u_g\right) + \frac{\partial(\rho w^*)}{\rho \partial Z} = 0.$$

Using (14.40) and (14.41) we finally obtain

$$\frac{\partial u_{ag}^*}{\partial X} + \frac{\partial v_{ag}^*}{\partial Y} + \frac{\partial(\rho w^*)}{\rho \partial Z} = 0. \tag{14.50}$$

Thus, even after introducing the new independent variables (X, Y, Z) and the new ageostrophic components $(u_{ag}^*, v_{ag}^*, w^*)$ the form of the continuity equation is unaltered. This is indeed remarkable.

Let us now find the relationship between q_g and Φ. Since $q_g = \frac{g}{\rho\theta_0} J \frac{\partial\theta}{\partial Z}$ we can use the hydrostatic equation to write

$$J^{-1} = \frac{1}{\rho q_g}\Phi_{ZZ}. \tag{14.51}$$

The product of (14.46) and (14.47) minus the square of (14.48) yields

$$J^2\left[\left(1 - \frac{1}{f^2}\Phi_{XX}\right)\left(1 - \frac{1}{f^2}\Phi_{YY}\right) - \frac{1}{f^4}\Phi_{XY}^2\right] = J. \tag{14.52}$$

Substituting (14.52) into (14.51) for J^{-1} we obtain

$$\frac{1}{f^2}\left(\Phi_{XX} + \Phi_{YY}\right) - \frac{1}{f^4}\left(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2\right) + \frac{1}{\rho q_g}\Phi_{ZZ} = 1. \tag{14.53}$$

To summarize we now collect (14.40), (14.41), (14.38), (14.50), (14.43) and (14.53) into a complete system of equations in the eight unknowns $u_g, v_g, \theta, \Phi, q_g, u_{ag}^*, v_{ag}^*, w^*$, all of which are functions of (X, Y, Z, T):

$$\frac{\partial u_g}{\partial T} + u_g\frac{\partial u_g}{\partial X} + v_g\frac{\partial u_g}{\partial Y} - f v_{ag}^* = 0, \tag{14.54}$$

$$\frac{\partial v_g}{\partial T} + u_g\frac{\partial v_g}{\partial X} + v_g\frac{\partial v_g}{\partial Y} + f u_{ag}^* = 0, \tag{14.55}$$

$$\left(f v_g, -f u_g, \frac{g}{\theta_0}\theta\right) = \left(\frac{\partial\Phi}{\partial X}, \frac{\partial\Phi}{\partial Y}, \frac{\partial\Phi}{\partial Z}\right), \tag{14.56}$$

$$\frac{\partial u_{ag}^*}{\partial X} + \frac{\partial v_{ag}^*}{\partial Y} + \frac{\partial(\rho w^*)}{\rho \partial Z} = 0, \tag{14.57}$$

$$\frac{\partial\theta}{\partial T} + u_g\frac{\partial\theta}{\partial X} + v_g\frac{\partial\theta}{\partial Y} + \frac{\theta_0}{g}q_g\rho w^* = 0, \tag{14.58}$$

$$\frac{1}{f^2}\left(\Phi_{XX} + \Phi_{YY}\right) - \frac{1}{f^4}\left(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2\right) + \frac{1}{\rho q_g}\Phi_{ZZ} = 1. \tag{14.59}$$

Formally these equations are almost identical to the quasi-geostrophic equations (see Table 14.1). The differences are as follows: (1) the independent variables are X, Y, Z, T; (2) the ageostrophic flow is u_{ag}^*, v_{ag}^*, w^*; (3) the effective static stability is ρq_g, rather than N^2, which is a standard atmosphere static stability.

14.3 Ageostrophic circulations

Although the form of the semi-geostrophic equations (14.54)–(14.59) is not very convenient for computation, it is convenient for derivation of the ageostrophic diagnostic equation. Actually this equation can take two forms, which we now derive. Because of the similarity in the forms of the semi-geostrophic and quasi-geostrophic equations the analysis here follows closely that leading from (13.44)–(13.46) to (13.57).

Taking $f(\partial/\partial Z)$ of (14.54) and (14.55), and taking $\partial/\partial X$ and $\partial/\partial Y$ of (14.58) we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial T}+\mathbf{v}_g\cdot\nabla_X\right)f\frac{\partial u_g}{\partial Z}-f^2\frac{\partial v^*_{ag}}{\partial Z} &= -f\frac{\partial \mathbf{v}_g}{\partial Z}\cdot\nabla_X u_g \\ &= -\frac{g}{\theta_0}\left(\mathbf{k}\times\nabla_X\theta\right)\cdot\left(\mathbf{k}\times\frac{\partial\mathbf{v}_g}{\partial Y}\right) \\ &= -\frac{g}{\theta_0}\frac{\partial\mathbf{v}_g}{\partial Y}\cdot\nabla_X\theta = Q_2 \end{aligned} \tag{14.60}$$

$$\begin{aligned}\left(\frac{\partial}{\partial T}+\mathbf{v}_g\cdot\nabla_X\right)f\frac{\partial v_g}{\partial Z}+f^2\frac{\partial u^*_{ag}}{\partial Z} &= -f\frac{\partial \mathbf{v}_g}{\partial Z}\cdot\nabla_X v_g \\ &= \frac{g}{\theta_0}\left(\mathbf{k}\times\nabla_X\theta\right)\cdot\left(\mathbf{k}\times\frac{\partial\mathbf{v}_g}{\partial X}\right) \\ &= \frac{g}{\theta_0}\frac{\partial\mathbf{v}_g}{\partial X}\cdot\nabla_X\theta = -Q_1 \end{aligned} \tag{14.61}$$

$$\left(\frac{\partial}{\partial T}+\mathbf{v}_g\cdot\nabla_X\right)\frac{g}{\theta_0}\frac{\partial\theta}{\partial X}+\frac{\partial}{\partial X}\left(q_g\rho w^*\right) = -\frac{g}{\theta_0}\frac{\partial\mathbf{v}_g}{\partial X}\cdot\nabla_X\theta = Q_1 \tag{14.62}$$

$$\left(\frac{\partial}{\partial T}+\mathbf{v}_g\cdot\nabla_X\right)\frac{g}{\theta_0}\frac{\partial\theta}{\partial Y}+\frac{\partial}{\partial Y}\left(q_g\rho w^*\right) = -\frac{g}{\theta_0}\frac{\partial\mathbf{v}_g}{\partial Y}\cdot\nabla_X\theta = Q_2. \tag{14.63}$$

Here we have used the thermal wind equation $f(\partial\mathbf{v}_g/\partial Z) = (g/\theta_0)\mathbf{k}\times\nabla_X\theta$ and the relations $\nabla_X u_g = \mathbf{k}\times(\partial\mathbf{v}_g/\partial Y)$, $\nabla_X v_g = -\mathbf{k}\times(\partial\mathbf{v}_g/\partial X)$. Q_1 and Q_2 are the components of the vector $\mathbf{Q}$, defined by

$$\mathbf{Q} = (Q_1, Q_2) = -\frac{g}{\theta_0}\left(\frac{\partial\mathbf{v}_g}{\partial X}\cdot\nabla_X\theta, \frac{\partial\mathbf{v}_g}{\partial Y}\cdot\nabla_X\theta\right). \tag{14.64}$$

Subtracting (14.61) from (14.62), adding (14.60) and (14.63), and using the thermal wind equation, we can write our system of diagnostic equations for the ageostrophic flow as

$$\frac{\partial}{\partial X}\left(q_g\rho w^*\right)-f^2\frac{\partial u^*_{ag}}{\partial Z} = 2Q_1, \tag{14.65}$$

$$\frac{\partial}{\partial Y}\left(q_g\rho w^*\right)-f^2\frac{\partial v^*_{ag}}{\partial Z} = 2Q_2, \tag{14.66}$$

$$\frac{\partial u^*_{ag}}{\partial X}+\frac{\partial v^*_{ag}}{\partial Y}+\frac{\partial(\rho w^*)}{\rho\partial Z} = 0. \tag{14.67}$$

Adding $\partial/\partial X$ of (14.65) to $\partial/\partial Y$ of (14.66) and using the continuity equation (14.67) we obtain

$$\nabla^2_X\left(\rho q_g w^*\right)+f^2\frac{\partial}{\partial Z}\left(\frac{\partial(\rho w^*)}{\rho\partial Z}\right) = 2\nabla_X\cdot\mathbf{Q}, \tag{14.68}$$

which is the generalized omega equation. Compare this with the quasi-geostrophic omega equation (13.57). We note that the physical discussion following (13.57) carries over with little modification.

Another way of proceeding from (14.65)–(14.67) is to define the vector streamfunction $\mathbf{\Psi} = (\psi_1, \psi_2)$ such that

$$\rho u^*_{ag} = -\frac{\partial\psi_1}{\partial Z},\;\; \rho v^*_{ag} = -\frac{\partial\psi_2}{\partial Z},\;\; \rho w^* = \frac{\partial\psi_1}{\partial X}+\frac{\partial\psi_2}{\partial Y}. \tag{14.69}$$

The continuity equation (14.67) is then automatically satisfied. Equations (14.65) and (14.66) can then be written

$$\frac{\partial}{\partial X}\left(q_g\frac{\partial\psi_1}{\partial X}\right)+f^2\frac{\partial}{\partial Z}\left(\frac{\partial\psi_1}{\rho\partial Z}\right)=2Q_1-\frac{\partial}{\partial X}\left(q_g\frac{\partial\psi_2}{\partial Y}\right), \tag{14.70}$$

$$\frac{\partial}{\partial Y}\left(q_g\frac{\partial\psi_2}{\partial Y}\right)+f^2\frac{\partial}{\partial Z}\left(\frac{\partial\psi_2}{\rho\partial Z}\right)=2Q_2-\frac{\partial}{\partial Y}\left(q_g\frac{\partial\psi_1}{\partial X}\right), \tag{14.71}$$

which is an alternative to (14.68). In fact, (14.68) is easily derived from (14.69)–(14.71).

Thus we have separate equations for the circulation in the (X, Z) and (Y, Z) planes with vertical velocity terms providing a linkage in the form of the second terms on the right-hand sides. Scaling arguments suggest that the linkage term in the (X, Z) equation is negligible if the geostrophic length scale in the Y direction is much larger than the Rossby radius of deformation. The (X, Z) circulation equation then reduces to the cross-frontal circulation equation of Eliassen (1962). Thus (14.70) and (14.71) are the natural extension of Eliassen's equation to the three-dimensional domain, and not necessarily to frontal regions. Indeed, with the modifications noted previously, these equations are applicable in the quasi-geostrophic context also.

14.4 Comparison of semi-geostrophic and quasi-geostrophic theories

Hoskins (1975) has made the following comparison of the semi-geostrophic and quasi-geostrophic theories. The semi-geostrophic equations include the advection of an approximation to the full potential vorticity, as opposed to the quasi-potential vorticity advected in the quasi-geostrophic equations. Ageostrophic advection of potential vorticity and potential temperature is included in the former system. In quasi-geostrophic theory the only ageostrophic advection is by the vertical velocity where it acts on a standard vertical temperature gradient.

From this point on, we simplify the comparison by considering only the uniform potential vorticity case. The quasi-geostrophic equations would be identical with the semi-geostrophic equations (14.54)–(14.59) except that Φ, X, Y and Z would be replaced by ϕ, x, y and z.

The more important difference is that the geostrophic velocities and potential temperature are predicted at (X, Y, Z) not (x, y, z). From the nature of the coordinate transformation it is easily seen (e.g., Fig. 14.1) that positive relative vorticity is increased and the region in which it occurs is decreased. Negative relative vorticity is decreased in magnitude and the region in which it occurs is increased. Thus the semi-geostrophic theory allows the production of sharp fronts, small vigorous low pressure cells, and broad weak high pressure cells. This clearly depends on the inclusion of advection by the convergent or divergent wind field and the nonlinearity in the stretching of vorticity. Using the semi-geostrophic equations, systems that are vertical using quasi-geostrophic theory tend to orient themselves along absolute vortex lines [from (14.33)]. This was commented on by Fjortoft. This is exactly the sloping of frontal regions found in the frontal studies.

Another property of nonlinear baroclinic waves as described by the semi-geostrophic equations may be simply inferred. The phases of the temperature and pressure waves as given by quasi-geostrophic theory are always such that near the surface, the temperature perturbation maximum occurs in the cyclonic region and the minimum in the anticyclonic region. Thus the semi-geostrophic equations imply that the area of warm anomaly is decreased and that of cold anomaly is increased. Higher up in the atmosphere the reverse is true. This is clearly the occlusion process in which warm air is moved upward, thus releasing potential energy. As remarked previously, in quasi-geostrophic theory, potential energy is released by moving warm air poleward and the occlusion process is not described.

Despite the much less stringent approximations made in the derivation of the semi-geostrophic equations, they predict merely a distortion of the quasi-geostrophic solution in a range of parameter space in which the derivation of the latter is not consistent. This may go some way to explaining the point commented on earlier: that the quasi-geostrophic equations have been successfully used in situations in which their validity is not clear.

Table 14.1: A Comparison of Equations

Primitive Equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv_{ag} = 0$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu_{ag} = 0$$
$$g\frac{\theta}{\theta_0} = \frac{\partial \phi}{\partial z}$$
$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial (\rho w)}{\rho \partial z} = 0$$
$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0$$

Geostrophic Momentum Approximation:

$$\frac{\partial u_g}{\partial t} + u\frac{\partial u_g}{\partial x} + v\frac{\partial u_g}{\partial y} + w\frac{\partial u_g}{\partial z} - fv_{ag} = 0$$
$$\frac{\partial v_g}{\partial t} + u\frac{\partial v_g}{\partial x} + v\frac{\partial v_g}{\partial y} + w\frac{\partial v_g}{\partial z} + fu_{ag} = 0$$
$$g\frac{\theta}{\theta_0} = \frac{\partial \phi}{\partial z}$$
$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial (\rho w)}{\rho \partial z} = 0$$
$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0$$

Quasi-Geostrophic Equations:

$$\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} - fv_{ag} = 0$$
$$\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} + fu_{ag} = 0$$
$$g\frac{\theta}{\theta_0} = \frac{\partial \phi}{\partial z}$$
$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial (\rho w)}{\rho \partial z} = 0$$
$$\frac{\partial \theta}{\partial t} + u_g\frac{\partial \theta}{\partial x} + v_g\frac{\partial \theta}{\partial y} + \frac{\theta_0}{g}N^2 w = 0$$

Semi-Geostrophic Equations:

$$\frac{\partial u_g}{\partial T} + u_g\frac{\partial u_g}{\partial X} + v_g\frac{\partial u_g}{\partial Y} - fv_{ag}^* = 0$$
$$\frac{\partial v_g}{\partial T} + u_g\frac{\partial v_g}{\partial X} + v_g\frac{\partial v_g}{\partial Y} + fu_{ag}^* = 0$$
$$g\frac{\theta}{\theta_0} = \frac{\partial \Phi}{\partial Z}$$
$$\frac{\partial u_{ag}^*}{\partial X} + \frac{\partial v_{ag}^*}{\partial Y} + \frac{\partial (\rho w^*)}{\rho \partial Z} = 0$$
$$\frac{\partial \theta}{\partial T} + u_g\frac{\partial \theta}{\partial X} + v_g\frac{\partial \theta}{\partial Y} + \frac{\theta_0}{g}\rho q_g w^* = 0$$