(5.) $$f(x,y)=\frac{x^2y-x^2y\cos(x^2+y^2)}{(x^2+y^2)^3}=$$

$$=\frac{x^2y}{x^2+y^2}\cdot\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

$t=x^2+y^2$

$(x,y)\to(0,0) \Rightarrow t\to 0$

$$\lim_{t\to 0}\frac{1-\cos t}{t^2}=\lim_{t\to 0}\frac{(1-\cos t)'}{(t^2)'}=\lim_{t\to 0}\frac{\overbrace{\sin t}^{\to 1}}{2t}=\frac{1}{2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}=\frac{1}{2}$$

$$0\le|f(x,y)|=\underbrace{\frac{x^2}{x^2+y^2}}_{\le 1}\cdot\overbrace{|y|}^{\to 0}\cdot\overbrace{\left|\frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}\right|}^{\to\frac{1}{2}}\to 0$$

teorem o sendviču

$$\Rightarrow \lim_{(x,y)\to(0,0)}|f(x,y)|=0 \Rightarrow \lim_{(x,y)\to(0,0)}f(x,y)=0 \quad (*)$$

$$\tilde{f}(x,y)=\begin{cases}\dfrac{x^2y-x^2y\cos(x^2+y^2)}{(x^2+y^2)^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$\tilde{f}$ je neprekidna u $(0,0)$ zbog $(*)$

u svim ostalim točkama je $\tilde{f}$ neprekidna kao kvocijent neprekidnih f-ja.

6.

$$f(x,y)=\frac{\sin(x^2+3y^2)}{\ln(1+2x^2+y^2)}=\frac{\sin(x^2+3y^2)}{x^2+3y^2}\cdot(x^2+3y^2)\cdot\frac{2x^2+y^2}{\ln(1+2x^2+y^2)}\cdot\frac{1}{2x^2+y^2}$$

$$=\frac{\sin(x^2+3y^2)}{x^2+3y^2}\cdot\frac{2x^2+y^2}{\ln(1+2x^2+y^2)}\cdot\frac{x^2+3y^2}{2x^2+y^2}$$

$(x,y)\to(0,0) \Rightarrow x^2+3y^2\to 0 \quad 2x^2+y^2\to 0$

$$\lim_{t\to 0}\frac{\sin t}{t}=1 \qquad \lim_{t\to 0}\frac{t}{\ln(1+t)}=\lim_{t\to 0}\frac{t'}{(\ln(1+t))'}=$$

$$=\lim_{t\to 0}\frac{1}{\frac{1}{1+t}}=1$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}\frac{\sin(x^2+3y^2)}{x^2+3y^2}\cdot\frac{2x^2+y^2}{\ln(1+2x^2+y^2)}=1$$

Očito $\lim_{(x,0)\to(0,0)}\frac{x^2+3\cdot 0^2}{2x^2+0^2}=\frac{1}{2} \Rightarrow \lim_{(x,0)\to(0,0)} f(x,y)=1\cdot 1\cdot\frac{1}{2}=\frac{1}{2}$

$\lim_{(0,y)\to(0,0)}\frac{0^2+3y^2}{2\cdot 0^2+y^2}=3 \Rightarrow \lim_{(0,y)\to(0,0)} f(x,y)=1\cdot 1\cdot 3=3$

$\frac{1}{2}\neq 3 \Rightarrow$ Ne postoji $\lim_{(x,y)\to(0,0)} f(x,y)$ pa se

f ne može proširiti do neprekidne u $(0,0)$

7.

$$f(x,y)=\begin{cases}\dfrac{x^5-x^2y^3}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$f(x,y)=\frac{x^5}{(x^2+y^2)^2}-\frac{x^2y^3}{(x^2+y^2)^2}=$$

$$=\left(\frac{x^2}{x^2+y^2}\right)^2\cdot x-\frac{x^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}\cdot y$$

$$0\leq\left|\left(\frac{x^2}{x^2+y^2}\right)^2\cdot x\right|=\underbrace{\left(\frac{x^2}{x^2+y^2}\right)^2}_{\leq 1}\cdot\underbrace{|x|}_{\to 0}\to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}\left(\frac{x^2}{x^2+y^2}\right)^2\cdot x=0 \qquad (1)$$

$$0\leq\underbrace{\left|\frac{x^2}{x^2+y^2}\right|}_{\leq 1}\cdot\underbrace{\left|\frac{y^2}{x^2+y^2}\right|}_{\leq 1}\cdot\underbrace{|y|}_{\to 0}\leq|y|\to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}\cdot y=0 \qquad (2)$$

$1\&(2)\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)=0 \Rightarrow f$ je neprekidna u $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^5}{t^4} - 0}{t} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Jedini kandidat za diferencijal je lin. op. s matricom

$$\left[\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0)\right] = \left[1 \quad 0\right]$$

$$T(x,y) = \left[1 \quad 0\right] \cdot \begin{bmatrix} x \\ y \end{bmatrix} = x \quad \text{je lin. op.}$$

Provjerimo je li $\lim_{(x,y)\to(0,0)} \frac{\|f(x,y)-f(0,0)-T(x,y)\|}{\|(x,y)\|} = 0$

$$g(x,y) = \frac{\|f(x,y)-f(0,0)-T(x,y)\|}{\|(x,y)\|} = \frac{\left|\frac{x^5-x^2y^3}{(x^2+y^2)^2} - x\right|}{\sqrt{x^2+y^2}}$$

$$= |x| \cdot \frac{\left|\frac{x^4-xy^3}{(x^2+y^2)^2} - \frac{(x^2+y^2)^2}{(x^2+y^2)^2}\right|}{\sqrt{x^2+y^2}} =$$

$$= \frac{|x|}{\sqrt{x^2+y^2}} \cdot \left|\frac{-xy^3-2x^2y^2-y^4}{(x^2+y^2)^2}\right|$$

$$= \frac{|x|}{\sqrt{x^2+y^2}} \cdot \frac{|y^2(xy+2x^2+y^2)|}{(x^2+y^2)^2}$$

$$\lim_{(\frac{1}{n},\frac{1}{n})\to(0,0)} g\left(\frac{1}{n},\frac{1}{n}\right) = \frac{\cancel{\frac{1}{n}}}{\cancel{\frac{1}{n}}\sqrt{2}} \cdot \frac{\cancel{\frac{1}{n^2}\cdot 4\cdot\frac{1}{n^2}}}{\cancel{4\cdot\frac{1}{n^4}}} = \frac{1}{\sqrt{2}}$$

$$\lim_{(\frac{1}{n},0)\to(0,0)} g\left(\frac{1}{n},0\right) = 0$$

$$0 \neq \frac{1}{\sqrt{2}} \Rightarrow \text{Ne postoji } \lim_{(x,y)\to(0,0)} \frac{\|f(x,y)-f(0,0)-T(x,y)\|}{\|(x,y)\|}$$

pa f nije diferencijabilna na cijelom $\mathbb{R}^2$ jer nije diferencijabilna u 0.