Zadatak 2.4.2. Razvijte u Maclaurinov red funkciju f zadanu sa $f(z) = e^z \sin z$.

Rješenje.

$$\begin{aligned} f(z) = e^z \sin(z) = e^z \frac{e^{iz} - e^{-iz}}{2i} &= \frac{1}{2i}\left(e^{z(1+i)} - e^{z(1-i)}\right) \\ &= \frac{1}{2i}\left(\sum_{n=0}^{+\infty} \frac{(z\cdot(1+i))^n}{n!} - \sum_{n=0}^{+\infty} \frac{(z\cdot(1-i))^n}{n!}\right) \\ &= \sum_{n=0}^{+\infty} \frac{(1+i)^n - (1-i)^n}{2i\cdot n!} z^n. \end{aligned}$$

Primijetimo da vrijedi

$$(1\pm i)^n = \left(\sqrt{2}\left(\cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4}\right)\right)^n = \sqrt{2^n}\left(\cos\frac{n\pi}{4} \pm i\sin\frac{n\pi}{4}\right)$$

pa imamo

$$\frac{(1+i)^n - (1-i)^n}{2i} = \sqrt{2^n}\sin\frac{n\pi}{4}.$$

Kada to uvrstimo u gornji izraz za $f(z)$, dobijemo

$$f(z) = \sum_{n=0}^{+\infty} \frac{\sqrt{2^n}\sin\frac{n\pi}{4}}{n!} z^n.$$

□

Zadatak 2.4.3. Razvijte u Maclaurinov red:

(a) $\dfrac{1}{1+z+z^2+z^3}$

(b) $\dfrac{z}{(z^2+1)(z^2-4)}$

(c) $\ln(z^2 - 3z + 2)$

Rješenje. (a) Kao u (b) dijelu zadatka 2.3.2. za $z \neq 1$ imamo

$$\begin{aligned} \frac{1}{1+z+z^2+z^3} = \frac{1-z}{1-z^4} &= (1-z)\frac{1}{1-z^4} \\ &= (1-z)\sum_{n=0}^{+\infty} z^{4n} = \sum_{n=0}^{+\infty} z^{4n} - \sum_{n=0}^{+\infty} z^{4n+1} \\ &= 1 - z + z^4 - z^5 + z^8 - z^9 + \dots \end{aligned}$$

(b)

$$\begin{aligned} \frac{z}{(z^2+1)(z^2-4)} = z\,\frac{1}{(z^2+1)(z^2-4)} &= z\cdot\frac{(z^2+1)-(z^2-4)}{(z^2+1)(z^2-4)}\cdot\frac{1}{5} \\ &= \frac{z}{5}\left(\frac{1}{z^2-4} - \frac{1}{z^2+1}\right) = \frac{z}{5}\left(-\frac{1}{4}\cdot\frac{1}{1-\left(\frac{z}{2}\right)^2} - \frac{1}{1+z^2}\right) \\ &= \frac{z}{5}\left(-\frac{1}{4}\cdot\sum_{n=0}^{+\infty}\frac{z^{2n}}{2^{2n}} - \sum_{n=0}^{+\infty}(-1)^n z^{2n}\right) \\ &= \sum_{n=0}^{+\infty}\frac{1}{5}\left((-1)^{n+1} - 4^{-(n+1)}\right) z^{2n+1} \end{aligned}$$

(c)

$$
\begin{aligned}
\ln(z^2-3z+2) &= \ln(z^2-z-2z+2) = \ln((z-1)(z-2)) = \ln((1-z)(2-z)) \\
&\overset{\diamond}{=} \ln(1-z) + \ln\left(2\left(1-\frac{z}{2}\right)\right) = \ln(1-z) + \ln 2 + \ln\left(1-\frac{z}{2}\right) \\
&= \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{(-z)^n}{n} + \ln 2 + \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\left(-\frac{z}{2}\right)^n}{n} \\
&= \ln 2 + \sum_{n=1}^{+\infty}(-1)^{n-1}\left(\frac{(-z)^n + \left(-\frac{z}{2}\right)^n}{n}\right) \\
&= \ln 2 + \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\left((-1)^n + \left(-\frac{1}{2}\right)^n\right) z^n \\
&= \ln 2 + \sum_{n=1}^{+\infty}\left(-\frac{1}{n}\right)\left(1+\frac{1}{2^n}\right) z^n
\end{aligned}
$$

Za $\diamond$ pogledajte napomenu u vježbama nakon zadatka 2.4.3.

□

Zadatak 2.4.4. Razvijte u Maclaurinov red funkciju f zadanu sa $f(z) = \sqrt{z+i}$. Pritom za vrijednost funkcije uzmite glavnu vrijednost korijena,

$$
\sqrt{z} = \sqrt{|z|}\left(\cos\frac{\arg z}{2} + i\sin\frac{\arg z}{2}\right).
$$

Rješenje.

$$
\begin{aligned}
\sqrt{z+i} &= \sqrt{i}\sqrt{1-iz} = \left[\sqrt{i} = \frac{1+i}{\sqrt{2}}\right] \\
&= \frac{1+i}{\sqrt{2}}\sqrt{1-iz} = \left[(1+s)^\alpha = \sum_{n=0}^{+\infty}\binom{\alpha}{n}s^n,\ \text{za } |s|<1\right] \\
&= \frac{1+i}{\sqrt{2}}\sum_{n=0}^{+\infty}\binom{1/2}{n}(-1)^n i^n z^n
\end{aligned}
$$

□