Evidence for a $k^{-5/3}$ Law Inertial Range in Mesoscale Two-Dimensional Turbulence

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(Manuscript received 6 October 1978, in final form 28 February 1979)

ABSTRACT

The observational evidence for $k^{-5/3}$ law behavior in the atmospheric kinetic energy spectrum is reviewed. This evidence includes the results of atmospheric wind variability studies and the observed scale dependence of atmospheric dispersion. It is concluded that $k^{-5/3}$ law behavior for time and space scales greater than those that can be three-dimensionally isotropic is probably a manifestation of the two-dimensional reverse-cascading energy inertial range.

1. Introduction

Several theoretical models have been used to describe the behavior of atmospheric turbulence. These include the three-dimensional inertial range model ($k^{-5/3}$ law), the two-dimensional energy inertial range model ($k^{-5/3}$ law), and the two-dimensional enstrophy inertial range model (k^{-3} law). The turbulence models listed above have been used in theoretical investigations of the predictability of atmospheric motions (Lorenz, 1969; Leith and Kraichnan, 1972). They also form the basis for parameterizing subgrid-scale motions in numerical prediction models (Kraichnan, 1976; Basdevant *et al.*, 1978). Whereas the different models lead to different results in predictability studies, subgrid-scale modeling and in describing atmospheric transport, it is important to determine, as precisely as possible, the range of atmospheric scales to which each model is relevant.

The three-dimensional inertial range theory is the original theoretical model considered by Kolmogoroff (1941) to describe the statistical small-scale behavior of locally isotropic, homogeneous turbulence. Although the relevance of three-dimensional turbulence theory to the atmosphere has been the subject of much debate, it now appears that its relevance is limited to spatial scales typically on the order of 100 m or less. Under disturbed conditions such as in convective storms where the vertical velocity component has considerable energy the three-dimensional inertial range model may be relevant out to scales on the order of 1–10 km. Nevertheless, numerous studies of atmospheric spectra have shown a $k^{-5/3}$ law behavior out to much larger scales. Ellsaesser (1969a), for example, applied three-dimensional turbulence theory to obtain estimates of the eddy dissipation rate from an analysis of the variability of synoptic-scale winds. This approach was justified by Ellsaesser on the grounds that the wind variability σ_τ followed a $\tau^{1/3}$ power law consistent with the $k^{-5/3}$ law. Also, scale-dependent diffusivity $K \propto L^{4/3}$ (Richardson, 1926) and dispersion $\overline{x^2} \propto t^3$ (Bauer, 1974; Gifford, 1957, 1977) are consistent with $k^{-5/3}$ law inertial range behavior.

Two-dimensional turbulence theory has attracted widespread interest among meteorologists in the past decade following the work of Kraichnan (1967) and others. Although Kraichnan discussed two formal inertial ranges: a $k^{-5/3}$ law energy cascading range and a k^{-3} law enstrophy cascading range, most attention has been focused on the k^{-3} law enstrophy range appropriate for geostrophic turbulence (Charney, 1971; Tennekes, 1978). Several authors have provided evidence for the existence of the k^{-3} law in large-scale atmospheric spectra ($\lambda > 1000$ km) (Wiin-Nielsen, 1967; Julian *et al.*, 1970; Desbois, 1975). Also, Morel and Larcheveque (1974) have reported indirect evidence that the relative dispersion of Eole balloon pairs at 200 mb in the Southern Hemisphere is consistent with a k^{-3} law spectra down to 100 km.

Thus, it would appear that there is substantial evidence for the relevance of the $k^{-5/3}$ law three-dimensional inertial range on the microscale and the k^{-3} law two-dimensional enstrophy inertial range on the macroscale. Evidence will be presented here to substantiate $k^{-5/3}$ law behavior for intermediate scales and it will be suggested that this behavior is consistent with the existence of a mesoscale, two-dimensional, reverse-cascading energy inertial range.

2. The variability of mesoscale winds and the $k^{-5/3}$ law

The variability of the wind has been related to turbulence theory by Ellsaesser (1969b). Temporal wind variability, the variability of wind with time at a fixed point and altitude, has been the subject of most wind variability research. It can be defined as

$$\sigma_\tau = \overline{\{[v(t) - v(t + \tau)]^2\}}^{1/2}, \quad (1)$$

where τ is the lag time. For stationary turbulence the velocity structure function $D_v(\tau)$ is given by

$$D_v(\tau) = \sigma_\tau{}^2 = 2\overline{(v')^2}[1 - R(\tau)], \quad (2)$$

where $R(\tau)$ is the Eulerian autocorrelation function defined by

$$R(\tau) = \frac{\overline{v'(t)v'(t + \tau)}}{\overline{(v')^2}} \quad (3)$$

and $v' = v - \bar{v}$.

In the spatial domain the $k^{-5/3}$ law takes the form

$$E(k) \propto \epsilon^{2/3}k^{-5/3}, \quad (4)$$

where E is the spectral density, ϵ the eddy dissipation rate per unit mass, and k the wavenumber ($k = 2\pi/\lambda$). The $k^{-5/3}$ law of Eq. (4) is of the same form in two and three dimensions. For some purposes the structure function is more convenient, i.e.,

$$\left.\begin{aligned} D_l(r) &= A_l\epsilon^{2/3}r^{2/3} \\ D_t(r) &= A_t\epsilon^{2/3}r^{2/3} \end{aligned}\right\}, \quad (5)$$

where D_l and D_t are the Eulerian structure functions for the longitudinal and transverse components of velocity, respectively. According to theory A_l and A_t in Eq. (5) are universal constants. A_l is close to 1.75. Kolmogoroff (1941) demonstrated that for an incompressible fluid in three dimensions

$$A_t = 4A_l/3. \quad (6)$$

In two dimensions a similar derivation yields

$$A_t = 5A_l/3, \quad (7)$$

which was recognized by Hutchings (1955).

By employing Taylor's transformation, it is possible to relate space and time variations. Thus, if $r = \bar{v}\tau$ is used in Eq. (5) we find

$$\left.\begin{aligned} D_l(\tau) &= A_l(\bar{v})^{2/3}\epsilon^{2/3}\tau^{2/3} \\ D_t(\tau) &= A_t(\bar{v})^{2/3}\epsilon^{2/3}\tau^{2/3} \end{aligned}\right\}. \quad (8)$$

It follows that the temporal variability (which is proportional to the square root of the structure function) of both horizontal components of velocity will follow a $\tau^{1/3}$ power law provided Taylor's transformation is valid. Also, the ratios of variabilities of the transverse and longitudinal components of velocity should be close to 1.165 ($=\sqrt{4/3}$) and 1.29 ($=\sqrt{5/3}$) in three and two dimensions, respectively.

Taylor's transformation plays an important role in relating frequency and wavenumber spectra. Its use is widespread in micrometeorology and in wind tunnel studies of turbulence. Hutchings (1955), Pinus (1968) and Ellsaesser (1969a) have even used the Taylor transformation for studies of synoptic-scale turbulence.

The validity of the Taylor transformation requires the advection of turbulent structure past a sensor with the velocity $\bar{v}$ of the mean flow. The time $\tau_a = \lambda/\bar{v}$ it takes an eddy of length scale λ to be advected past the sensor must be much less than the decay time $\tau_d \sim \lambda[\overline{(v')^2}]^{-1/2}$ for the eddy. This requirement implies that $[\overline{(v')^2}]^{1/2} \ll \bar{v}$.

Experimentally, the Taylor transformation can be tested by comparing simultaneous measurements of temporal and spatial correlation functions. This has been done with positive results for microscale turbulence in the wind tunnel and in the atmospheric boundary layer. The same experiment cannot easily be done on the mesoscale especially in the free atmosphere.

Ogura (1953) and Gifford (1956) have further considered the validity of Taylor's transformation. They found that if it is assumed the spatial structure function obeys an $r^{2/3}$ law the temporal structure function will be proportional to τ^m with $2/3 < m < 1$ according to how large $[\overline{(v')^2}]^{1/2}$ is relative to $\bar{v}$. This leads to an m only slightly greater than $2/3$ for normal atmospheric conditions when $[\overline{(v')^2}]^{1/2} < \bar{v}$ (Ellsaesser, 1969a). Thus, with some reservations, it appears reasonable to use the Taylor transformation on the mesoscale.

On the synoptic scale there is reason to suspect the validity of the Taylor transformation. For large-scale atmospheric motion Kao and Wendell (1970) and Kao *et al.* (1970) found markedly different power law dependence for the wavenumber and frequency spectra. One explanation for the failure of the Taylor transformation on the synoptic scale is that the disturbance kinetic energy is largely due to Rossby waves which move with a wavelength-dependent phase velocity and are thus not simply advected by the mean flow.

3. Evidence for the existence of a $k^{-5/3}$ law two-dimensional inertial range

Several authors have studied synoptic-scale wind variability and atmospheric kinetic energy spectra in the light of turbulence theory. Hutchings (1955) considered the variability of wind in two sets of data: 500 mb radar (balloon) winds for Larkhill, England, taken at 6 h intervals from December 1944 to February 1945, and 300 mb radar (balloon) winds for Auckland, New Zealand, for the winter period from June to August 1952. The first set of data yielded wind variability for both zonal and meridional com-

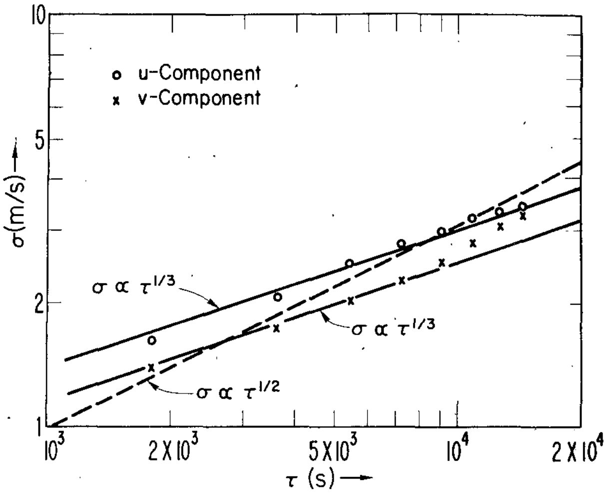

FIG. 1. Average u and v component wind variability σ(m s^{-1}) from high-resolution balloon soundings. Variabilities averaged over fifty 100 m intervals. Power laws of $\tau^{1/3}$ and $\tau^{1/2}$ are shown for reference (after Gage and Jasperson, 1978).

ponents very consistent with the $\tau^{1/3}$ power law out to lags of 36 h. The second set of data yielded $\tau^{0.44}$ for the lag variability of both wind components. Pinus (1968), employing Taylor's transformation, determined wavenumber spectra of synoptic-scale atmospheric kinetic energy over the Moscow area by means of a time series analysis of routine upper wind observations spaced 6 h apart during the period March 1965–March 1966. The data were analyzed by season for 11 altitudes in the range 0–20 km. The resulting spectra exhibited a $k^{-5/3}$ law inertial range behavior over much of the wavenumber range.

Mesoscale wind variability has recently been studied below 5 km using accurate high-resolution wind soundings in Minnesota (Gage and Jasperson, 1979). Fig. 1 shows the lag variability obtained during a 7 h experiment on 31 March 1976 in which balloons were launched at 30 min intervals. The variability data plotted in Fig. 1 is the mean variability obtained by averaging together the variabilities obtained for each wind component for fifty 100 m height intervals. The resulting mean wind variability for both wind components follows the $\tau^{1/3}$ law out to 150 min. With a mean wind of about 10 m s^{-1} this implies energy inertial range behavior out to ~100 km.

For the data set of 31 March the mean wind was nearly from the south. The ratio of west wind variability to south wind variability for this data set gives 1.19 (from 30 to 180 min lag) without correction for the departure of the south wind from the longitudinal direction. Correcting for the departure of the south wind from the direction of the mean wind increases the ratio and makes it more consistent with the value 1.29 predicted for two-dimensional turbulence.

The variability of jet stream winds using VHF Doppler radar has been studied by Gage and Clark (1978). The radar sampled the south wind about once a minute on 15–16 April 1976 over an altitude range from 5 to 13 km during the passage of a polar front jet stream. Unfortunately, in order to sample the south wind as rapidly as possible, the west wind was sampled only hourly. Fig. 2 reproduces the mean variability obtained by averaging the variabilities for each height. The mean variability follows the $\tau^{1/3}$ law out to at least 4 h in lag time. Since the mean wind was on the order of 30 m s^{-1}, this implies energy inertial range behavior out to horizontal distances on the order of 400 km or more. However, a systematic trend in the power law dependence for variabilities at individual heights makes the precision of agreement shown here somewhat fortuitous. Nevertheless, lag variability at altitudes in close proximity to the strong shear zones above and below the jet maxima was quite consistent with the $\tau^{1/3}$ law.

The dispersion of tracers in the atmosphere also supports the $k^{-5/3}$ law inertial range behavior in mesoscale atmospheric turbulence. The dispersion of particles for the $k^{-5/3}$ law inertial range is governed by $d\overline{x^2}/dt \propto x^{4/3}$ and $\overline{x^2} \propto t^3$. Here, $(\overline{x^2})^{1/2}$ is the cloud width and t is travel time. Gifford (1977) in a comprehensive review of tropospheric relative diffusion finds $\overline{x^2} \propto t^3$ out to 1–3000 s with a somewhat slower rate of diffusion out to 10^5 s.

Lin (1972) raised the possibility that the relative dispersion of particles could be consistent with the k^{-3} law inertial range for which $d\overline{x^2}/dt \propto \overline{x^2}$ and $\overline{x^2} \propto \exp(t)$. Although Lin (1972) and Morel and Larcheveque (1974) give evidence of this behavior from Eole data for travel times >1 day and for cloud widths >50 km, Gifford (1977) found no evidence

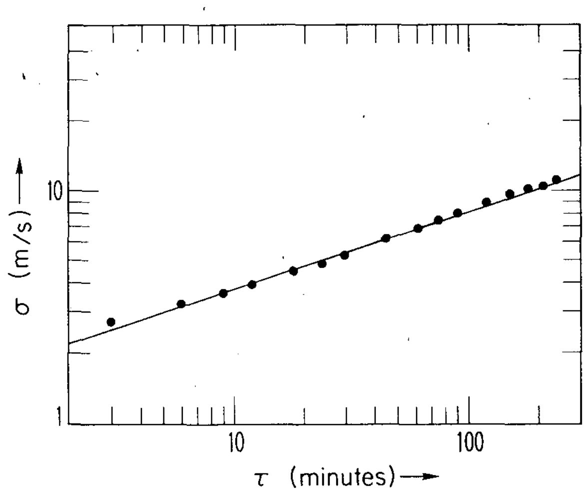

FIG. 2. Average variability of radar observed south wind during a jet stream passage. The power law $\tau^{1/3}$ is shown for reference (after Gage and Clark, 1978).

of an accelerated rate of diffusion in tropospheric data for travel times of up to one day and cloud widths of up to 100 km.

4. Discussion

Theoretical studies (Kraichnan, 1967; Leith, 1968; Lilly, 1969) of two-dimensional turbulence have shown that a source of energy and enstrophy isolated at wavenumber k_i leads to a wavenumber spectrum with a discontinuity at k_i. For $k < k_i$ energy is cascaded to lower wavenumbers and $E(k) \propto \epsilon^{2/3} k^{-5/3}$ and for $k > k_i$ enstrophy is cascaded to large wavenumbers and $E(k) \propto \eta^{2/3} k^{-3}$, where η is the enstrophy cascade rate. The net result is a spectrum with $k^{-5/3}$ behavior at the low-wavenumber end and k^{-3} behavior at the high-wavenumber end.

In applying the two-dimensional turbulence model to the atmosphere it is reasonable to consider two source regions in the wavenumber domain. On the macroscale at hemispheric wavenumbers near 6 ($\sim$4500 km) baroclinic instability provides a source of energy (and enstrophy). For hemispheric wavenumbers >10 ($\sim$3000 km), an approximate k^{-3} spectrum is found presumably associated with this source. A second source is found at small scales where microscale turbulence is generated. Microscale turbulence is usually thought to lose its energy to viscous dissipation by means of a three-dimensional cascade of energy into the high-wavenumber end of the spectrum. However, the existence of a $k^{-5/3}$ law inertial range extending to larger scales implies that some of the microscale turbulent kinetic energy is being cascaded to lower wavenumbers consistent with the reverse cascade of two-dimensional turbulence theory. According to this view, enstrophy is being cascaded down the spectrum from the macroscale source and energy is being cascaded up the spectrum from the microscale source.

The concept of an enstrophy cascading k^{-3} spectrum at low wavenumbers acting simultaneously with a reverse energy cascading $k^{-5/3}$ spectrum at high wavenumbers raises questions of consistency with two-dimensional turbulence theory. Kraichnan (1967) demonstrated that energy could not be cascaded through the k^{-3} spectrum and that enstrophy could not be cascaded through the $k^{-5/3}$ spectrum. It follows that there must be a sink of energy and enstrophy at the wavenumber (or wavenumber range) of transition between the two ranges. Otherwise, one or both of these spectral ranges cannot be true manifestations of the formal inertial ranges of two-dimensional turbulence theory.

In light of the above, it is of interest to consider what possible sink there may be for energy and enstrophy on the 1000 km scale which, it seems likely, separates the enstrophy and energy cascading ranges. A possible candidate would appear to be the extratropical cold-core vortices associated with the decaying stage of the cyclone life cycle. The energetics of these vortices is not well understood but it does appear that some of their kinetic energy is converted into available potential energy (Palmén and Newton, 1969). The substantiation of the possible role of the cold-core vortices in the realization of the inertial ranges discussed here must await further research.

5. Concluding remarks

In this paper the evidence for energy inertial range behavior at time and space scales which are too large to be isotropic in three dimensions has been summarized. The interpretation has been offered that the reason for the $k^{-5/3}$ law behavior at such scales is the existence of an energy cascading, two-dimensional inertial range. The two-dimensional, energy cascading inertial range is associated with the transfer of energy from small scales to large scales: the reverse of the traditional energy cascade in three dimensions. Thus, it is conceivable that microscale energy sources (e.g., wind shear, breaking waves, thunderstorms, etc.) could cascade energy both ways in wavenumber space: to smaller scales consistent with three-dimensional turbulence theory and to larger scales consistent with two-dimensional turbulence theory. According to this view, $k^{-5/3}$ law behavior could be expected to extend well into the mesoscale and possibly to 1000 km or more—especially when the small-scale energy sources are very pronounced.

Acknowledgments. I wish to thank Drs. J. Herring, T. E. VanZandt and J. Weinstock for helpful discussions.

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